Problem Set 7: Due Friday, February 21

Problem 1: Let

$$A = \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} : c \in \mathbb{R} \right\} \qquad \text{and} \qquad B = \left\{ \begin{pmatrix} 5 \\ 7 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} : c \in \mathbb{R} \right\}.$$

In this problem, we will work through the outline of how to show that $A = B$ via a double containment proof.

a. Let's show that $A \subseteq B$. Let $\vec{u} \in A$ be arbitrary. By definition of A, we can fix $c \in \mathbb{R}$ with

$$\vec{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

To show that $\vec{u} \in B$, we have to fill in the blank of

$$\vec{u} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} + \underline{\quad} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

with a real number. What real number works? Justify your choice.

b. Let's show that $B \subseteq A$. Let $\vec{u} \in B$ be arbitrary. By definition of B, we can fix $c \in \mathbb{R}$ with

$$\vec{u} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

To show that $\vec{u} \in A$, we have to fill in the blank of

$$\vec{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \underline{\quad} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

with a real number. What real number works? Justify your choice.

Problem 2: In each of the following cases, determine if the given function $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. If yes, explain why. If no, provide an explicit counterexample.

a. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 7y \\ 5x - 4y \end{pmatrix}$.

b. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} xy \\ x + y \end{pmatrix}$.

c. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \sin^2(x^3) + y \cos^2(x^3)) \\ y \end{pmatrix}$.

Problem 3: Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ -x + y \end{pmatrix}.$$

Plot the values of at least 4 points and where T sends them, and then use that to describe the action of T geometrically.

Problem 4: Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ -5x + 3y \end{pmatrix}.$$

Show that

$$\begin{pmatrix} -18 \\ 47 \end{pmatrix} \in \text{range}(T)$$

by explicitly finding $\vec{v} \in \mathbb{R}^2$ with

$$T(\vec{v}) = \begin{pmatrix} -18 \\ 47 \end{pmatrix}.$$

Problem 5: Show that the linear transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + 2y \\ 3x + 6y \end{pmatrix}$$

is not injective and not surjective.