Problem Set 5: Due Monday, February 10

Problem 1: Each of the following is an *attempted* description of a set, but some of them do not make sense. In each part, explain whether the definition is valid or not. When it is a valid description, give 3 specific elements of the corresponding set.
a. $\{n^2 + 19m : n, m \in \mathbb{N}\}$.
b. $\{x \in \mathbb{Q} : x^2 + 5x - 3\}$.
c. $\{n \in \mathbb{Z} : 5 < 2^n \text{ and } 10n < 81\}$.
d. $\{x^2 + 6 \in \mathbb{R} : x - 1 > 0\}$.

Problem 2: Let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - 8x$. Show that f is not injective.

Problem 3: Define a function $f \colon \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$ as follows. On input $n \in \{0, 1, 2, 3, 4\}$, let $f(n)$ be the remainder that arises when you divide the number $3n$ by 5. Is f injective, surjective, both, or neither? Explain.

Problem 4: Let $A = \{1, 2\}$. Give an example, with explanation, of two functions $f \colon A \to A$ and $g \colon A \to A$ such that $f \circ g \neq g \circ f$.

Problem 5: Consider the function $f \colon \mathbb{Q} \to \mathbb{Q}$ given by $f(a) = 3a + 2$. We clearly have $\text{range}(f) \subseteq \mathbb{Q}$ by definition. Thus, to show that $\mathbb{Q} = \text{range}(f)$, it suffices to show that $\mathbb{Q} \subseteq \text{range}(f)$. To do this, we need to show how to take an arbitrary $b \in \mathbb{Q}$, and fill in the blank in $f(\_\_\_\_) = b$ with an element of $\mathbb{Q}$. In this problem, we first do a few examples, and then handle a general b.
a. Fill in the blank in $f(\_\_\_\_) = 2$ with an element of $\mathbb{Q}$.
b. Fill in the blank in $f(\_\_\_\_) = -19$ with an element of $\mathbb{Q}$.
c. Fill in the blank in $f(\_\_\_\_) = 55$ with an element of $\mathbb{Q}$.
d. Let $b \in \mathbb{Q}$ be arbitrary. Fill in the blank in $f(\_\_\_\_) = b$ with an element of $\mathbb{Q}$ (your answer will depend on b), and justify that your choice works.

Problem 6: Determine if the three lines described by the equations $2x+y = 5$, $7x-2y = 1$, and $-5x+3y = 4$ intersect. That is, determine whether there exists a point that lies on all three of the corresponding lines. Explain your reasoning using a few sentences.