Problem Set 3: Due Monday, February 3

Problem 1: Consider the statement "If $a \in \mathbb{Z}$ is odd, then $4a+1$ is even". Your friend claims to have a proof, and presents the following argument:

Let $a \in \mathbb{Z}$ an arbitrary odd integer. Since a is odd, we can fix $m \in \mathbb{Z}$ with $a = 2m+1$. We then have that

$$\begin{aligned} 4a+1 &= 4 \cdot (2m+1) + 1 \\ &= 8m+5 \\ &= 2 \cdot \left(\frac{8m+5}{2}\right). \end{aligned}$$

We have shown the existence of an $n \in \mathbb{Z}$ with $4a+1 = 2n$, so $4a+1$ is even.

a. Pinpoint the error in your friend's argument. Be as specific as you can.
b. Is the statement in question true or false? Justify your answer carefully.

Problem 2: Consider the following statement: For all $a \in \mathbb{R}$, if $a < 1$, then $a^2 < 1$.
a. Write the negation of each of the the given statement so that no "not" appears.
b. Show that the given statement is false by arguing that its negation is true.
Note: Be very careful on part (a). Don't just mindlessly apply symbolical manipulations! Think through the logic carefully. What precisely would count as a counterexample to an "if...then..." statement? It might help to look at Problem 1 and consider the truth table at the top of p. 24 of the course notes.

Problem 3: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "If $a, b \in \mathbb{Z}$ are both odd, then $a+b$ is even". As in Problem Set 2, please write up the entire argument, not just the pieces that go into the blanks.

Let $a, b \in \mathbb{Z}$ be two arbitrary odd integers. Since a is odd, we can ________________. Since b is odd, we can ________________. Now notice that $a+b =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $a+b$ is an even integer.

Problem 4: Write both the converse and contrapositive of each of the following statements (no need to argue whether any of the them are true or false). In each case, get rid of all occurrences of *not* in the final result.
a. If $a \in \mathbb{Z}$ and $a \geq 2$, then $4a > 7$.
b. If $x, y \in \mathbb{R}$ and $x^4 + y^4 = 1$, then $x^2 + y^2 \leq 2$.
c. If $a \in \mathbb{Z}$ and there exists $m \in \mathbb{Z}$ with $a = 10m$, then there exists $m \in \mathbb{Z}$ with $a = 5m$.

Problem 5: Consider the following statement:

If $a \in \mathbb{Z}$ and $3a+5$ is even, then a is odd.

a. Write down the contrapositive of the given statement.
b. Show that the original statement is true by proving that the contrapositive is true.

Problem 6: Let $A = \{\sin x : x \in \mathbb{R}\}$.
a. In class, we talked about how we could always turn a parametric description of a set into our other description (by carving out of a bigger set) by using a "there exists" quantifier. Do that for our set A above.
b. Find another way to describe A by carving it out of a set using a property without any quantifiers. Briefly explain why your set is equal (no need to give a formal proof).