Problem Set 20: Due Monday, April 27

Problem 1: Define $T\colon \mathcal{P}_2 \to \mathbb{R}^2$ by letting

$$T(f) = \begin{pmatrix} f(0) \\ f(2) \end{pmatrix}.$$

It turns out that T is a linear transformation. Let $\alpha = (x^2, x, 1)$, which is a basis for $\mathcal{P}_2$.
a. Let

$$\varepsilon_2 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

be the standard basis of $\mathbb{R}^2$. What is $[T]_\alpha^{\varepsilon_2}$?
b. Let

$$\beta = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),$$

which is a basis of $\mathbb{R}^2$. What is $[T]_\alpha^\beta$?

Problem 2: Let V be the vector space of all 2×2 matrices. Define $T\colon V \to V$ by letting

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Notice that the function T takes an input matrix and outputs the result of switching the rows and columns (which is called the *transpose* of the original matrix). It turns out that T is a linear transformation. Let

$$\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

and recall that α is a basis for V. What is $[T]_\alpha^\alpha$? Explain briefly.

Problem 3: Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with

$$[T] = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 8 & 5 \end{pmatrix}.$$

Find, with explanation, a basis for range(T).

Problem 4: Let V be the vector space of all 2×2 matrices. Define $T\colon V \to \mathbb{R}$ by letting

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = 2a - d.$$

a. Show that T is a linear transformation.
b. Show that T is surjective.
c. Give an example, with proof, of $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \text{Null}(T)$ such that $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is linearly independent.