Problem Set 2: Due Friday, January 31

Note: In Problems 4 and 5, write up the whole proof, not just the pieces that go into the blanks. Also, fill in as much as necessary in the blanks so that no steps are omitted. Finally, if you would prefer to write the argument differently in your words, please feel free to do so.

Problem 1: Let $A = \{1,2,3,4\}$. Suppose that $P(x,y)$ and $Q(x,y)$ are expressed by the following tables:

P	1	2	3	4
1	F	T	T	F
2	F	T	F	T
3	T	F	F	F
4	F	F	F	F

Q	1	2	3	4
1	F	T	T	T
2	F	T	T	T
3	T	F	T	F
4	F	F	T	T

Interpret the diagrams as follows. If $x, y \in \{1,2,3,4\}$, to determine the truth value of $P(x,y)$, go to row x and column y. For example, $P(1,2)$ is true and $P(3,3)$ is false. Determine, with careful explanation, whether each of the following are true or false:
a. There exist $x, y \in A$ with $\text{Not}(x = y)$, $P(x,y)$, and $P(y,x)$.
b. For all $x, y \in A$, we have $\text{Not}(Q(x,y))$.
c. For all $x, y \in A$, if $P(x,y)$, then $Q(x,y)$.
d. For all $x, y \in A$, either $Q(x,y)$ or $Q(y,x)$.
e. For all $x \in A$, there exists $y \in A$ with $\text{Not}(P(x,y))$.
f. There exists $x \in A$ such that for all $y \in A$, we have $P(x,y)$.
g. There exists $y \in A$ such that for all $x \in A$, we have $Q(x,y)$.

Problem 2: Let A be the set of all people. Given $x, y \in A$, let $M(x,y)$ be true if x has ever sent a text message to y, and be false otherwise. Express each of the following statements carefully using only A and M, our quantifiers "for all" and "there exists", and the connectives "=" "and", "or", "not", and "if...then...". In other words, express each of the following like the statements in Problem 1.
a. Everybody has sent a text message to somebody.
b. Somebody has never received a text message from anyone.
c. Somebody has sent a text message to (at least) two different people.
d. Somebody has sent a text message to each person that they have received a message from.

Problem 3: Write the negation of each of the the following statements so that no "not" appears. You do *not* need to explain if the statements are true or false.
a. For all $x \in \mathbb{R}$, we have $e^x \neq 0$.
b. There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, we have $x + y^2 \geq 3$.
c. For all $x, y \in \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that both $x < q$ and $q < y$.
d. There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}^+$ with $x^n + y > 0$.

Problem 4: In this problem, we write a careful proof of a fact that we discussed on the second day of class. Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "For all $x, y \in \mathbb{Z}$, we have that $22x + 14y$ is even":

Let $x, y \in \mathbb{Z}$ be arbitrary. We have $22x + 14y =$ ________________. Since ________________$\in \mathbb{Z}$, we conclude that $22x + 14y$ is even.

Problem 5: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "For all $a \in \mathbb{Z}$, we have that $2a^3 + 6a - 3$ is odd":

Let $a \in \mathbb{Z}$ be arbitrary. We have $2a^3 + 6a - 3 =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $2a^3 + 6a - 3$ is odd.