Writing Assignment 6: Due Wednesday, March 13

Problem 1: Suppose that $\alpha = (\vec{u}_1, \vec{u}_2)$ and $\beta = (\vec{w}_1, \vec{w}_2)$ are both bases of $\mathbb{R}^2$. Show that there exists an invertible 2×2 matrix R such that $[\vec{v}]_\beta = R \cdot [\vec{v}]_\alpha$ for all $\vec{v} \in \mathbb{R}^2$. Moreover, explicitly describe how to calculate R, and be sure to argue that R is invertible.

Problem 2: Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and assume that $T \circ T = T$. Show that $\text{range}(T) \cap \text{Null}(T) = \{\vec{0}\}$.

Problem 3: Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is a function, and that $f(c \cdot x) = c \cdot f(x)$ for all $c, x \in \mathbb{R}$. Show that there exists an $r \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $f(x) = rx$.
Note: In this problem, we are looking at functions with domain and codomain $\mathbb{R}$ (rather than $\mathbb{R}^2$). The hypothesis is saying that f "preserves scalar multiplication", so long as you interpret a number as a 1-dimensional vector. Thus, this problem is saying that every function $f \colon \mathbb{R} \to \mathbb{R}$ that preserves scalar multiplication is one of the boring ones, like $f(x) = 2x$, $f(x) = -5x$, or $f(x) = \pi x$.