Problem Set 9: Due Monday, March 4

Problem 1: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of reflecting the plane across the line $2x+y=0$.
a. Calculate $[T]$.
b. Calculate $T\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right)$.

Problem 2: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first projecting $\vec{v}$ onto the line $y = 3x$, and then projecting the result onto the line $y = 4x$. Explain why T is a linear transformation, and then calculate $[T]$.

Problem 3: Let $\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$ be a nonzero vector. Recall that we defined the function $P_{\vec{w}}\colon \mathbb{R}^2 \to \mathbb{R}^2$ that projects points onto the line $\text{Span}(\vec{w})$. By Proposition 2.5.10, we know that $P_{\vec{w}}$ is a linear transformation, and that it has standard matrix

$$A = \begin{pmatrix} \frac{a^2}{a^2+b^2} & \frac{ab}{a^2+b^2} \\ \frac{ab}{a^2+b^2} & \frac{b^2}{a^2+b^2} \end{pmatrix}.$$

a. Show that $A \cdot A = A$ by simply computing it.
b. By interpreting the action of $P_{\vec{w}}$ geometrically, explain why you should expect that $A \cdot A = A$.
Cultural Aside: A matrix A that satisfies $A \cdot A = A$ is called *idempotent*.

Problem 4: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the point on the line $y = x + 1$ that is closest to $\vec{v}$. Is T is a linear transformation? Explain.

Problem 5: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first reflecting $\vec{v}$ across the x-axis, and then reflecting the result across the y-axis.
a. Compute $[T]$.
b. The action of T is the same as a certain rotation. Explain which rotation it is.

Problem 6: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $r \in \mathbb{R}$. We know from Proposition 2.4.7 that $r \cdot T$ is a linear transformation. Show that if

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$[r \cdot T] = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}.$$

In other words, if we define the multiplication of a matrix by a scalar as in Definition 2.6.3, then the standard matrix of $r \cdot T$ is obtained by multiplying every element of $[T]$ by r.