Problem Set 6: Due Friday, February 22

Problem 1: For each part, explain your reasoning using a sentence or two.
a. Find an example of a choice for $\vec{v}, \vec{u} \in \mathbb{R}^2$ such that the solution set to $-x + 9y = -6$ is $\{\vec{v} + t\vec{u} : t \in \mathbb{R}\}$.
b. Find an example of $\vec{u} \in \mathbb{R}^2$ such that the solution set of $5x + 3y = 0$ is $\text{Span}(\vec{u})$.
c. Find an example of a choice for $a, b, c \in \mathbb{R}$ such that the solution set of $ax + by = c$ is $\text{Span}\left(\begin{pmatrix} 2 \\ -7 \end{pmatrix}\right)$.

Problem 2:
a. Show that

$$\left(\begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$$

is a basis of $\mathbb{R}^2$.
b. Explicitly compute the unique values of $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} 12 \\ 7 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

c. Explicitly compute the unique values of $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} 13 \\ 25 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Problem 3: Find, with explanation, all values of $c \in \mathbb{R}$ such that

$$\left(\begin{pmatrix} c-3 \\ 1 \end{pmatrix}, \begin{pmatrix} 10 \\ c \end{pmatrix}\right)$$

is a basis of $\mathbb{R}^2$.

Problem 4: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement following statement: If $\vec{u}, \vec{w} \in \mathbb{R}^2$ and $\vec{w} \in \text{Span}(\vec{u})$, then $\text{Span}(\vec{w}) \subseteq \text{Span}(\vec{u})$. Please write out the whole proof, and either underline or highlight the parts that you are adding.

Let $\vec{v} \in \text{Span}(\vec{w})$ be arbitrary. By assumption, we know that $\vec{w} \in \text{Span}(\vec{u})$, so we can ________________. Since $\vec{v} \in \text{Span}(\vec{w})$, we can ________________. Now notice that $\vec{v} =$________________. Since ________________$\in \mathbb{R}$, we conclude that $\vec{v} \in \text{Span}(\vec{u})$.

Problem 5: Let $\vec{u}, \vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$, and suppose that $\vec{v}_1, \vec{v}_2 \in \text{Span}(\vec{u})$. Show that if $\vec{v}_1 \neq \vec{0}$, then $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$.
Note: You have to use the fact that $\vec{v}_1 \neq \vec{0}$ somewhere, so explicitly point out where you are using that assumption.

Problem 6: Define a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+5 \\ 2x-y \end{pmatrix}.$$

Determine, with explanation, whether T is a linear transformation.