Problem Set 4: Due Monday, February 11

Problem 1: Describe the set $\{x \in \mathbb{R} : x^2 - 3x - 4 > 0\}$ in another way by writing it as a union of two sets with simpler descriptions. Briefly explain why your set is equal.

Problem 2: Let $A = \{6n : n \in \mathbb{N}\} \cap \{10n : n \in \mathbb{N}\}$.
a. Write down the smallest 3 elements of A, and briefly explain how you determined them.
b. Make a conjecture about how to describe A parametrically (no need to prove this conjecture).

Problem 3: Given two sets A and B, we define

$$A \triangle B = \{x : x \text{ is an element of exactly one of } A \text{ or } B\},$$

and we call this set the *symmetric difference* of A and B. For example, we have

$$\{4, 5, 6, 8\} \triangle \{5, 6, 7, 8\} = \{4, 7\}.$$

a. Determine $\{1, 3, 8, 9\} \triangle \{2, 3, 4, 7, 8\}$.
b. Write $A \triangle B$ in terms of the sets A and B using only the operations of union, intersection, and relative complement (i.e. set difference). Explain.
c. What are the smallest 9 elements of the set $\{2n : n \in \mathbb{N}\} \triangle \{3n : n \in \mathbb{N}\}$?
d. Make a conjecture about how to write $\{2n : n \in \mathbb{N}\} \triangle \{3n : n \in \mathbb{N}\}$ as the union of 3 pairwise disjoint sets (no need to prove this conjecture, but do write the 3 sets parametrically).

Problem 4: Define $f \colon \mathbb{R} \to \mathbb{R}$ by letting $f(x) = e^x$. Write down a description of the set range(f) by carving it out of a set using a property without any quantifiers. Briefly explain why your set is equal (no need to give a formal proof).

Problem 5: Define $f \colon \mathbb{N}^+ \to \mathbb{N}^+$ by letting $f(n)$ be the number of positive divisors of n. Define $g \colon \mathbb{N}^+ \to \mathbb{N}$ by letting $g(n)$ be the number of primes less than or equal to n. For example, we have $g(1) = 0$, $g(2) = 1$, and $g(6) = 3$.
a. Calculate, with explanation, the values of $(g \circ f)(6)$ and $(g \circ f)(36)$.
b. Find an example, with explanation, of an $n \in \mathbb{N}^+$ with $(g \circ f)(n) = 3$.

Problem 6: Consider the function $f \colon \mathbb{Q} \to \mathbb{Q}$ given by $f(a) = 5a - 3$. We clearly have range$(f) \subseteq \mathbb{Q}$ by definition. Thus, to show that $\mathbb{Q} = \text{range}(f)$, it suffices to show that $\mathbb{Q} \subseteq \text{range}(f)$. To do this, we need to show how to take an arbitrary $b \in \mathbb{Q}$, and fill in the blank in $f(\_\_\_\_) = b$ with an element of $\mathbb{Q}$. In this problem, we first do a few examples, and then handle a general b.
a. Fill in the blank in $f(\_\_\_\_) = 7$ with an element of $\mathbb{Q}$.
b. Fill in the blank in $f(\_\_\_\_) = -53$ with an element of $\mathbb{Q}$.
c. Fill in the blank in $f(\_\_\_\_) = 1$ with an element of $\mathbb{Q}$.
d. Let $b \in \mathbb{Q}$ be arbitrary. Fill in the blank in $f(\_\_\_\_) = b$ with an element of $\mathbb{Q}$ (your answer will depend on b), and justify that your choice works.