Problem Set 3: Due Friday, February 8

Problem 1: Write both the converse and contrapositive of each of the following statements (no need to argue whether any of the them are true or false). In each case, get rid of all occurrences of *not* in the final result.
a. If $a \in \mathbb{Z}$ and $a \geq 2$, then $4a > 7$.
b. If $x, y \in \mathbb{R}$ and $x^4 + y^4 = 1$, then $x^2 + y^2 \leq 2$.
c. If $a \in \mathbb{Z}$ and there exists $m \in \mathbb{Z}$ with $a = 10m$, then there exists $m \in \mathbb{Z}$ with $a = 5m$.

Problem 2: Consider the following statement:

If $a \in \mathbb{Z}$ and $3a + 5$ is even, then a is odd.

a. Write down the contrapositive of the given statement.
b. Show that the original statement is true by proving that the contrapositive is true.

Problem 3: Let $A = \{\sin x : x \in \mathbb{R}\}$.
a. In class, we talked about how we could always turn a parametric description of a set into our other description (by carving out of a bigger set) by using a "there exists" quantifier. Do that for our set A above.
b. Find another way to describe A by carving it out of a set using a property without any quantifiers. Briefly explain why your set is equal (no need to give a formal proof).

Problem 4: Let $A = \{12n - 7 : n \in \mathbb{Z}\}$ and let $B = \{4n + 1 : n \in \mathbb{Z}\}$.
a. Show that $B \not\subseteq A$.
b. Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement that $A \subseteq B$.

Let $a \in A$ be arbitrary. By definition of A, we can _______________. Now notice that $a =$ _______________. Since _______________ $\in \mathbb{Z}$, we conclude that $a \in B$.

Problem 5: Let $A = \{x^2 + 5 : x \in \mathbb{R}\}$ and let $B = \{y \in \mathbb{R} : y \geq 5\}$. In this problem, we show that $A = B$ by doing a double containment proof.
a. Prove that $A \subseteq B$.
b. Notice that $5, 9, 10, 42 \in B$. Show that $5, 9, 10, 42 \in A$ by explicitly finding a value of $x \in \mathbb{R}$ with $x^2 + 5 = 5$, then finding a value of $x \in \mathbb{R}$ with $x^2 + 5 = 9$, etc.
c. Let $y \in B$ in arbitrary. Following the pattern that you see in part (b), what value of $x \in \mathbb{R}$ do you think will demonstrate that $y \in A$?
d. Using your idea from part (c), write a careful proof showing that $B \subseteq A$. At some point you will need to use the fact that elements of B are greater than or equal to 5, so be sure to point out where that is important!