Problem Set 2: Due Monday, February 4

Note: In Problems 3, 4, and 5, write up the whole proof, not just the pieces that go into the blanks. Also, fill in as much as necessary in the blanks so that no steps are omitted. Finally, if you would prefer to write the argument differently in your words, please feel free to do so.

Problem 1: Write the negation of each of the the following statements so that no "not" appears. You do *not* need to explain if the statements are true or false.
a. For all $x \in \mathbb{R}$, we have $e^x \neq 0$.
b. There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, we have $x + y^2 \geq 3$.
c. For all $x, y \in \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that both $x < q$ and $q < y$.
d. There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}^+$ with $x^n + y > 0$.

Problem 2: Consider the following statement: For all $a \in \mathbb{R}$, if $a < 1$, then $a^2 < 1$.
a. Write the negation of each of the the given statement so that no "not" appears.
b. Show that the given statement is false by arguing that its negation is true.
Note: Be very careful on part (a). Don't just mindlessly apply symbolical manipulations! Think through the logic carefully. What precisely would you need to show in order to demonstrate the statement is false?

Problem 3: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "If $a, b \in \mathbb{Z}$ are both odd, then $a + b$ is even":

Let $a, b \in \mathbb{Z}$ be two arbitrary odd integers. Since a is odd, we can ________________. Since b is odd, we can ________________. Now notice that $a + b =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $a + b$ is an even integer.

Problem 4: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "For all $a \in \mathbb{Z}$, we have that $2a^3 + 6a - 3$ is odd":

Let $a \in \mathbb{Z}$ be arbitrary. We have $2a^3 + 6a - 3 =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $2a^3 + 6a - 3$ is odd. Since $a \in \mathbb{Z}$ was arbitrary, the result follows.

Problem 5: In this problem, we write a careful proof of a fact that we discussed on the second day of class. Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "For all $x, y \in \mathbb{Z}$, we have that $14x + 8y$ is even":

Let $x, y \in \mathbb{Z}$ be arbitrary. We have $14x + 8y =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $14x + 8y$ is even.

Problem 6: Consider the statement "If $a \in \mathbb{Z}$ is odd, then $4a + 1$ is even". Your friend claims to have a proof, and presents the following argument:

Let $a \in \mathbb{Z}$ an arbitrary odd integer. Since a is odd, we can fix $m \in \mathbb{Z}$ with $a = 2m + 1$. We then have that

$$\begin{aligned} 4a + 1 &= 4 \cdot (2m + 1) + 1 \\ &= 8m + 5 \\ &= 2 \cdot \left(\frac{8m + 5}{2} \right) \end{aligned}$$

We have shown the existence of an $n \in \mathbb{Z}$ with $4a + 1 = 2n$, so $4a + 1$ is even.

a. Pinpoint the error in your friend's argument. Be as specific as you can.
b. Is the statement in question true or false? Justify your answer carefully.