Problem Set 18: Due Monday, April 22

Problem 1: Let

$$\alpha = \left(\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ 9 \end{pmatrix} \right).$$

a. Show that α is a basis of $\mathbb{R}^3$.
b. Determine

$$\left[\begin{pmatrix} 1 \\ 5 \\ -5 \end{pmatrix} \right]_\alpha.$$

Problem 2: Consider the following elements of $\mathcal{P}_3$:

- $f_1(x) = x^3$.
- $f_2(x) = x^3 + x^2$.
- $f_3(x) = x^3 + x^2 + x$.
- $f_4(x) = x^3 + x^2 + x + 1$.

Let $\alpha = (f_1, f_2, f_3, f_4)$.
a. Show that α is a basis of $\mathcal{P}_3$.
b. Let $g(x) = 3x^3 + 7x^2 + 7x - 2$. Determine $[g]_\alpha$.

Problem 3: Let $W = \{f \in \mathcal{P}_2 : f(1) = 0\}$. In Problem 2 on Writing Assignment 8, you showed that W is a subspace of $\mathcal{P}_2$. Let $\alpha = (f_1, f_2)$ where:

- $f_1(x) = x^2 - 1$.
- $f_2(x) = x - 1$.

We then have that $W = \text{Span}(f_1, f_2)$ (this might have been the choice you used for part (b) of that problem; see the solutions to Writing Assignment 8 for a proof).
a. Show that α is linearly independent, and hence is a basis of W.
b. Determine $\dim(W)$.
c. Let $g(x) = 3x^2 + 5x - 8$. Show that $g \in W$, and determine $[g]_\alpha$.

Problem 4: Let V be the vector space of all 2×2 matrices. Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V : b = c \right\}.$$

It can be checked that W is a subspace of V (no need to do this). Find a basis for W, and determine $\dim(W)$.

Problem 5: In Problem 2 on Problem Set 17, you showed that

$$\left(\begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -8 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} \right)$$

was linearly dependent. Use your work in that problem to find a basis (with explanation) for the following subspace of $\mathbb{R}^4$:

$$W = \text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -8 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix}\right).$$