Written Assignment 7 : Due Wednesday, April 13

Problem 1: Let $a, b, c \in \mathbb{R}$ and let

$$M = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

a. Show that $\det(M) = (b-a)(c-a)(c-b)$.
b. Explain why M is invertible exactly when a, b, c are all distinct from each other.

Problem 2: Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix.
a. Show that $Col(AB) \subseteq Col(A)$, i.e. show that if $\mathbf{v} \in Col(AB)$, then $\mathbf{v} \in Col(A)$.
b. Show that $rank(AB) \leq rank(A)$.
c. Show that if B is an invertible $n \times n$ matrix (so $p = n$), then $rank(AB) = rank(A)$.
Hint for c: You can get the reverse inequality using a clever application of part b.

Problem 3: Let P be an $n \times n$ stochastic matrix (remember this means that all entries of P are nonnegative and each column sums to 1). In class, we outlined a very sophisticated argument that there exists a probability vector $\mathbf{q}$ with $P\mathbf{q} = \mathbf{q}$. In this problem we prove the weaker statement that there exists a nonzero vector $\mathbf{x}$ with $P\mathbf{x} = \mathbf{x}$.
a. Show that if you add up the rows of $P - I$, you get the zero vector.
b. Show that the rows of $P - I$ are linearly dependent.
c. Show that $rank(P - I) \leq n - 1$.
d. Show that $Nul(P - I) \neq \{\mathbf{0}\}$.
e. Show that there exists a nonzero vector $\mathbf{x}$ with $P\mathbf{x} = \mathbf{x}$.