Written Assignment 5 : Due Wednesday, March 9

Note: When we use the word *or* in mathematics, we mean that at least one of the options happens (so not necessarily exactly one). In other words, when we say "A or B", we are allowing the possibility that both A and B are true. Be sure to interpret *or* in this way in both Problem 1b and Problem 2b.

Problem 1: Let V be a vector space. Prove each of the following. For this problem, be very explicit and mention which of the vector space axioms you are using in each step of your argument.
a. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Show that $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \mathbf{w} + (\mathbf{u} + \mathbf{v})$.
b. Suppose that $c \in \mathbb{R}$ and $\mathbf{u} \in V$ are such that $c \cdot \mathbf{u} = \mathbf{0}$. Show that either $c = 0$ or $\mathbf{u} = \mathbf{0}$.

Problem 2: Let V be a vector space. Suppose that H and K are both subspaces of V.
a. Let $H \cap K$ be the intersection of H and K, i.e. $H \cap K = \{\mathbf{v} \in V : \mathbf{v} \in H \text{ and } \mathbf{v} \in K\}$. Show that $H \cap K$ is a subspace of V.
b. Let $H \cup K$ be the union of H and K, i.e. $H \cup K = \{\mathbf{v} \in V : \mathbf{v} \in H \text{ or } \mathbf{v} \in K\}$. By constructing an explicit counterexample, show that $H \cup K$ need *not* be a subspace of V.
c. Let $H + K = \{\mathbf{v} \in V : \text{There exists } \mathbf{u} \in H \text{ and } \mathbf{w} \in K \text{ with } \mathbf{v} = \mathbf{u} + \mathbf{w}\}$. That is, $H + K$ is the set of all vectors in V which can be written as the sum of an element of H and an element of K. Show that $H + K$ is a subspace of V.

Problem 3: Let V be a vector space. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, and suppose that $\mathbf{u} \notin \mathit{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. Working directly from the the definition of linear independence (i.e. don't use an equivalent characterization), show that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}\}$ is linearly independent.