Written Assignment 4 : Due Wednesday, March 2

Problem 1: Recall that an arbirary function $f\colon A \to B$ is called *onto* if for every $b \in B$, there exists $a \in A$ with $f(a) = b$. Suppose that $f\colon A \to B$ and and $g\colon B \to C$ are both onto functions. Show, using this definition, that the composition $g \circ f\colon A \to C$ is onto. Make sure you explain any equations you write and any symbols you introduce.

Problem 2: An $n \times n$ matrix A is called *idempotent* if $A^2 = A$. For example, the zero matrix and the identity matrix are idempotent. More interesting examples are:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

a. Show that the only $n \times n$ idempotent matrix which is invertible is the identity matrix I.
b. Show that if A is idempotent, then $I - A$ is idempotent.
c. Show that if A is idempotent, then $I + A$ is invertible and $(I + A)^{-1} = I - \frac{1}{2}A$.
Note: Be very careful. It is not in general true that $(A + B)^2 = A^2 + 2AB + B^2$. Just any algebraic manipulation you use.

Problem 3: In this problem, we determine which 2×2 matrices commute with *every* 2×2 matrix.
a. Show that if $r \in \mathbb{R}$ and we let

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

then $AB = BA$ for every 2×2 matrix B.
b. Suppose that A is a 2×2 matrix which the property that $AB = BA$ for every 2×2 matrix B. Show that there exists $r \in \mathbb{R}$ such that

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

Hint: For part b, make strategic choices for B to make your life as simple as possible. I suggest thinking about matrices with lots of zeros.