Writing Assignment 6: Due Wednesday, October 12

Problem 1: Suppose that $\alpha = (\vec{u}_1, \vec{u}_2)$ and $\beta = (\vec{w}_1, \vec{w}_2)$ are both bases of $\mathbb{R}^2$. Show that there exists an invertible 2×2 matrix R such that $[\vec{v}]_\beta = R \cdot [\vec{v}]_\alpha$ for all $\vec{v} \in \mathbb{R}^2$. Explicitly describe how to calculate R, and be sure to argue that R is invertible.

Problem 2: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose that $ad - bc = 0$ and at least one of a, b, c, d is nonzero. Show that if $\vec{w} \in \mathsf{Null}(T)$ and $\vec{w} \neq \vec{0}$, then $\mathsf{Null}(T) = \mathsf{Span}(\vec{w})$.
Note: We know from Theorem 2.7.3 that there exists $\vec{u} \in \mathbb{R}^2$ with $\mathsf{Null}(T) = \mathsf{Span}(\vec{u})$. But this problem is asking you to show something stronger: if you take *any* nonzero vector in $\mathsf{Null}(T)$, then $\mathsf{Null}(T)$ is the span of that element.
Hint: You can use problems on past assignments to avoid doing a lot of hard work.

Problem 3: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Show that $\mathsf{Null}(T) \neq \{\vec{0}\}$ if and only if there exists a basis α of $\mathbb{R}^2$ such that the second column of $[T]_\alpha$ is the zero vector.
Note: Recall that "if and only if" means that each of the two directions are true, so you need to argue both.