Problem Set 8: Due Friday, September 30

Problem 1: For each of following, consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that has the given matrix as its standard matrix. Describe the action of T geometrically. It may help to plug in a few points and/or make some case distinctions.

a. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b. $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ for a fixed $k \in \mathbb{R}$ with $k > 0$.

c. $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ for a fixed $k \in \mathbb{R}$ with $k > 0$.

Problem 2: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of reflecting the plane across the line $2x+y=0$.

a. Calculate $[T]$.

b. Calculate $T\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right)$.

Problem 3: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first projecting $\vec{v}$ onto the line $y = 3x$, and then projecting the result onto the line $y = 4x$. Explain why T is a linear transformation, and then calculate $[T]$.

Problem 4: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the point on the line $y = x + 1$ that is closest to $\vec{v}$. Is T is a linear transformation? Explain.

Problem 5: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $r \in \mathbb{R}$. We know from Proposition 2.4.7 that $r \cdot T$ is a linear transformation. Show that if

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$[r \cdot T] = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}.$$

In other words, if we define the multiplication of a matrix by a scalar as in Definition 2.6.3, then the standard matrix of $r \cdot T$ is obtained by multiplying every element of $[T]$ by r.

Problem 6: Let A be a 2×2 matrix. Verify each of the following using the formula for the matrix-vector product.

a. $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$ for all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$.

b. $A(c \cdot \vec{v}) = c \cdot A\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$ and all $c \in \mathbb{R}$.

Note: Since matrices encode linear transformations, you should expect these to be true. In fact, we can argue that they are true by interpreting the matrix as being the standard matrix of a certain linear transformation, and then just appealing to the fact that linear transformation preserve addition and scalar multiplication. However, in this problem, you should just work through the computations directly.