Problem Set 20: Due Friday, December 2

Problem 1: Let V be the vector space of all 2×2 matrices. Define $T\colon V \to V$ by letting

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Notice that the function T takes an input matrix and outputs the result of switching the rows and columns (which is called the *transpose* of the original matrix). It turns out that T is a linear transformation. Let

$$\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

and recall that α is a basis for V. What is $[T]_\alpha^\alpha$? Explain briefly.

Problem 2: Consider the unique linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ with

$$[T] = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -2 \end{pmatrix}.$$

a. Find (with explanation) bases for each of $\mathsf{range}(T)$ and $\mathsf{Null}(T)$.
b. Calculate $\mathsf{rank}(T)$ and $\mathsf{nullity}(T)$.

Problem 3: Consider the unique linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}$ with

$$[T] = \begin{pmatrix} 0 & 1 & -3 & 7 \end{pmatrix}.$$

a. Find (with explanation) bases for each of $\mathsf{range}(T)$ and $\mathsf{Null}(T)$.
b. Calculate $\mathsf{rank}(T)$ and $\mathsf{nullity}(T)$.

Problem 4: Define $T\colon \mathcal{P}_5 \to \mathcal{P}_5$ by letting $T(f) = f''$, i.e. $T(f)$ is the second derivative of f. Determine, with explanation, both $\mathsf{rank}(T)$ and $\mathsf{nullity}(T)$.

Problem 5: Let V be the vector space of all 2×2 matrices. Explain why there is no surjective linear transformation $T\colon V \to \mathcal{P}_4$.