Problem Set 19: Due Monday, November 28

Problem 1: Working in $\mathbb{R}^4$, find the dimension of the subspace

$$W = \mathsf{Span}\left(\begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -6 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ 2 \\ 2 \end{pmatrix} \right).$$

Problem 2: Working in the vector space $\mathcal{F}$ of all functions $f \colon \mathbb{R} \to \mathbb{R}$, define the following:

- $f_1(x) = 2^x$.
- $f_2(x) = 3^x$.

Find, with explanation, the dimension of the subspace $W = \mathsf{Span}(f_1, f_2)$.

Problem 3: Define $T \colon \mathcal{P}_1 \to \mathbb{R}^2$ by letting

$$T(a + bx) = \begin{pmatrix} a - b \\ b \end{pmatrix}.$$

Show that T is a linear transformation.

Problem 4: Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the function

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \begin{pmatrix} x - y \\ x + z \\ y + z \end{pmatrix}.$$

a. Explain why T is a linear transformation.
b. Give an example of a nonzero $\vec{v} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{0}$.
c. Show that T is not injective.

Problem 5: Define $T \colon \mathcal{P}_2 \to \mathbb{R}^2$ by letting

$$T(f) = \begin{pmatrix} f(0) \\ f(2) \end{pmatrix}.$$

It turns out that T is a linear transformation. Let $\alpha = (x^2, x, 1)$, which is a basis for $\mathcal{P}_2$.
a. Let

$$\varepsilon_2 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

be the standard basis of $\mathbb{R}^2$. What is $[T]_\alpha^{\varepsilon_2}$?
b. Let

$$\beta = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),$$

which is a basis of $\mathbb{R}^2$. What is $[T]_\alpha^\beta$?