Problem Set 11: Due Monday, October 10

Problem 1: Consider the unique linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix}.$$

Let $\alpha = (\vec{u}_1, \vec{u}_2)$ where

$$\vec{u}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \text{and} \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In this problem, we compute $[T]_\alpha$ directly from the definition.
a. Show that $\alpha = (\vec{u}_1, \vec{u}_2)$ is a basis of $\mathbb{R}^2$.
b. Determine $T(\vec{u}_1)$ and then use this to compute $[T(\vec{u}_1)]_\alpha$.
c. Determine $T(\vec{u}_2)$ and then use this to compute $[T(\vec{u}_2)]_\alpha$.
d. Using parts (b) and (c), determine $[T]_\alpha$.

Problem 2: With the same setup as Problem 1, compute $[T]_\alpha$ using Proposition 3.2.6.

Problem 3: Again, use the same setup as in Problem 1. Let

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

In this problem, we compute $[T(\vec{v})]_\alpha$ in two different ways.
a. First determine $T(\vec{v})$, and then use this to compute $[T(\vec{v})]_\alpha$.
b. First determine $[\vec{v}]_\alpha$, and then multiply the result by your matrix $[T]_\alpha$ to compute $[T(\vec{v})]_\alpha$.

Problem 4: Consider the unique linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}.$$

Let $\alpha = (\vec{u}_1, \vec{u}_2)$ where

$$\vec{u}_1 = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \quad \text{and} \quad \vec{u}_2 = \begin{pmatrix} 9 \\ 4 \end{pmatrix}.$$

Compute $[T]_\alpha$ using any method.

Problem 5: Let $id\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the identity function, i.e. $id(\vec{v}) = \vec{v}$ for all $\vec{v} \in \mathbb{R}^2$. Show that

$$[id]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for every basis α of $\mathbb{R}^2$.

Problem 6: Let $a, b \in \mathbb{R}$ with at least one of a or b nonzero. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by projecting a point onto the line

$$\mathsf{Span}\left(\begin{pmatrix} a \\ b \end{pmatrix}\right).$$

In Proposition 2.5.10, we computed that

$$[T] = \begin{pmatrix} \frac{a^2}{a^2+b^2} & \frac{ab}{a^2+b^2} \\ \frac{ab}{a^2+b^2} & \frac{b^2}{a^2+b^2} \end{pmatrix}.$$

using dot products and algebra. Here, we compute $[T]$ in a different way by first computing $[T]_\alpha$ for a well-chosen basis α. Let $\alpha = (\vec{u}_1, \vec{u}_2)$ where

$$\vec{u}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \vec{u}_2 = \begin{pmatrix} -b \\ a \end{pmatrix}.$$

Notice that α is a basis of $\mathbb{R}^2$ because $a^2 + b^2 > 0$, as at least one of a or b is nonzero.

a. Without using our known formula for $[T]$, and doing very few (if any) computations, explain why

$$[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

b. Now use Proposition 3.2.6 to compute $[T]$ from $[T]_\alpha$.