

Problem Set 10: Due Friday, October 7

Problem 1: In this problem, let 0 denote the 2×2 zero matrix, i.e. the 2×2 matrix where all four entries are 0 .

- Give an example of a nonzero 2×2 matrix A with $A \cdot A = 0$.
- Show that if A is a 2×2 matrix and $A \cdot A = 0$, then A is not invertible.

Hint: Suppose that you are working in the world of real numbers. How would you convince yourself that if $x^2 = 0$, then x would have to be 0 ? If $x \neq 0$, what can you do to both sides?

Problem 2: Find a 2×2 matrix A with

$$A \cdot A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hint: Working computationally is not advised. Think about a linear transformation that you can apply to the plane so that when you do it twice, you end up sending $(1, 0)$ to $(0, 1)$ and sending $(0, 1)$ to $(-1, 0)$.

Problem 3: Let A, B, C all be invertible 2×2 matrices. Must there exist a 2×2 matrix X with

$$A(X + B)C = I?$$

Either justify carefully or give a counterexample.

Hint: Again, consider working in the world of numbers. If you have nonzero real numbers a, b , and c , must there exist a real number x with $a(x + b)c = 1$? If so, what is it? And how would you justify that it works?

Problem 4: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the unique linear transformation with

$$[T] = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}.$$

Explain why T has an inverse and calculate

$$T^{-1} \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix} \right).$$

Problem 5: Consider the following system of equations:

$$\begin{aligned} x + 4y &= -3 \\ 2x + 5y &= 8. \end{aligned}$$

- Rewrite the above system in the form $A\vec{v} = \vec{b}$ for some matrix A and vector \vec{b} .
- Explain why A is invertible and calculate A^{-1} .
- Use A^{-1} to solve the system.

Problem 6: Let

$$\alpha = \left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix} \right).$$

- Show that α is a basis for \mathbb{R}^2 .
- Compute $\left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} \right]_{\alpha}$.
- Compute $\left[\begin{pmatrix} 8 \\ 17 \end{pmatrix} \right]_{\alpha}$.

In each part, briefly explain how you carried out your computation.