Problem Set 21: Due Friday, December 2

Problem 1: Define $T\colon \mathcal{P}_2 \to \mathbb{R}^2$ by letting

$$T(f) = \begin{pmatrix} f(0) \\ f(2) \end{pmatrix}.$$

It turns out that T is a linear transformation. Let $\alpha = (x^2, x, 1)$, which is a basis for $\mathcal{P}_2$.
a. Let

$$\varepsilon_2 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

be the standard basis of $\mathbb{R}^2$. What is $[T]_\alpha^{\varepsilon_2}$?
b. Let

$$\beta = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),$$

which is a basis of $\mathbb{R}^2$. What is $[T]_\alpha^\beta$?

Problem 2: Let V be the vector space of all 2×2 matrices. Define $T\colon V \to V$ by letting

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Notice that the function T takes an input matrix and outputs the result of switching the rows and columns (which is called the *transpose* of the original matrix). It turns out that T is a linear transformation. Let

$$\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

and recall that α is a basis for V. What is $[T]_\alpha^\alpha$? Explain briefly.

Problem 3: Working in $\mathbb{R}^4$, let

$$W = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \\ 7 \end{pmatrix}, \begin{pmatrix} -4 \\ -7 \\ 0 \\ -3 \end{pmatrix} \right).$$

Find, with explanation, a basis for W and also $\dim(W)$.

Problem 4: Consider the unique linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ with

$$[T] = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -2 \end{pmatrix}.$$

a. Find bases for each of $\operatorname{range}(T)$ and $\operatorname{Null}(T)$.
b. Calculate $\operatorname{rank}(T)$ and $\operatorname{nullity}(T)$.

Problem 5: Define $T\colon \mathcal{P}_5 \to \mathcal{P}_5$ by letting $T(f) = f''$, i.e. $T(f)$ is the second derivative of f. Determine, with explanation, both $\operatorname{rank}(T)$ and $\operatorname{nullity}(T)$.