Problem Set 14: Due Monday, October 31

Problem 1: Let $V = \mathbb{R}^3$, but with the following operations:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$c \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \end{pmatrix}.$$

Show that there is no element of V that serves as $\vec{0}$. That is, show that there does not exist $\vec{z} \in V$ such that $\vec{v} + \vec{z} = \vec{v}$ for all $\vec{v} \in V$.

Problem 2: Let $V = \mathbb{R}^2$, but with the following operations:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

and

$$c \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ca_1 \\ a_2 \end{pmatrix}.$$

Also, let

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Show that V is not a vector space by explicitly finding a counterexample to one of the 10 properties.

Problem 3: Let V be a vector space. Show that $\vec{u} + (\vec{v} + \vec{w}) = \vec{w} + (\vec{v} + \vec{u})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$. Carefully state what property you are using in every step of your argument.

Problem 4: Let V be a vector space. Recall that, given $\vec{v} \in V$, we defined $-\vec{v}$ to be the unique $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$. Moreover, given $\vec{v}, \vec{w} \in V$, we defined $\vec{v} - \vec{w}$ to mean $\vec{v} + (-\vec{w})$. Prove each of the following, and carefully state what property and/or result you are using in every step of your arguments.
a. Show that $-(\vec{v} + \vec{w}) = (-\vec{v}) + (-\vec{w})$ for all $\vec{v}, \vec{w} \in V$.
b. Show that $c \cdot (\vec{v} - \vec{w}) = c \cdot \vec{v} - c \cdot \vec{w}$ for all $\vec{v}, \vec{w} \in V$ and all $c \in \mathbb{R}$.

Problem 5: Show that

$$\left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.