Written Assignment 3: Due Wednesday, October 1

Problem 1: Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

Is T injective? Justify your answer carefully.

Problem 2: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Recall that

$$\text{range}(T) = \{\vec{w} \in \mathbb{R}^2 : \text{There exists } \vec{v} \in \mathbb{R}^2 \text{ with } \vec{w} = T(\vec{v})\}.$$

Notice that $\vec{0} \in \text{range}(T)$ because we know that $T(\vec{0}) = \vec{0}$ by Proposition 2.14.
a. Show that if $\vec{w}_1, \vec{w}_2 \in \text{range}(T)$, then $\vec{w}_1 + \vec{w}_2 \in \text{range}(T)$.
b. Show that if $\vec{w} \in \text{range}(T)$ and $c \in \mathbb{R}$, then $c\vec{w} \in \text{range}(T)$.

Problem 3: We defined linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, but we can also define them from $\mathbb{R}$ to $\mathbb{R}$ as follows. A linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is a function $f\colon \mathbb{R} \to \mathbb{R}$ with both of the following properties:

- $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
- $f(c \cdot x) = c \cdot f(x)$ for all $c, x \in \mathbb{R}$.

Given any $r \in \mathbb{R}$, it is straightforward to check that the function $g_r\colon \mathbb{R} \to \mathbb{R}$ given by $g_r(x) = rx$ is a linear transformation (no need to do this). Show that these are the only linear transformations from $\mathbb{R}$ to $\mathbb{R}$. In other words, show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a linear transformation, then there exists $r \in \mathbb{R}$ with $f = g_r$.