Problem Set 9: Due Monday, October 6

Problem 1: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first projecting $\vec{v}$ onto the line $y = 3x$, and then projecting the result onto the line $y = 4x$. Explain why T is a linear transformation, and then calculate $[T]$.

Problem 2: Let

$$A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

a. Show that $A \cdot A = A$ by simply computing it.
b. By interpreting the action of A geometrically, explain why you should expect that $A \cdot A = A$.

Problem 3: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the point on the line $y = x + 1$ that is closest to $\vec{v}$. Is T is a linear transformation? Explain.

Problem 4: Let $\vec{w} \in \mathbb{R}^2$ be nonzero, and let $W = Span(\vec{w})$. Define $F_{\vec{w}}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $F_{\vec{w}}(\vec{v})$ be the result of reflecting $\vec{v}$ across the line W. Show that $F_{\vec{w}}$ is a linear transformation and determine the standard matrix $[F_{\vec{w}}]$.
Hint: Make use of projections. How can you determine $F_{\vec{w}}(\vec{v})$ using $\vec{v}$ and $P_{\vec{w}}(\vec{v})$?

Problem 5: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first reflecting $\vec{v}$ across the x-axis, and then reflecting the result across the y-axis.
a. Compute $[T]$.
b. The action of T is the same as a certain rotation. Explain which rotation it is.

Problem 6: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $r \in \mathbb{R}$. We know from Proposition 2.20 that $r \cdot T$ is a linear transformation. Show that if

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$[r \cdot T] = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

In other words, if we define the multiplication of a matrix by a scalar as in Definition 2.34, then the standard matrix of $r \cdot T$ is obtained by multiplying every element of $[T]$ by r.