Problem Set 2: Due Monday, September 8

Note: In Problems 2, 3, and 4, write up the whole proof, not just the pieces that go into the blanks. Also, fill in as much as necessary in the blanks so that no steps are omitted.

Problem 1: Write the negation of each of the the following statements so that no "not" appears. You do *not* need to explain if the statements are true or false (for now).
a. For all $x \in \mathbb{R}$, we have $e^x \neq 0$.
b. There exists $m, n \in \mathbb{Z}$ with $4m + 6n = 7$.
c. There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, we have $x + y^2 \geq 3$.
d. For all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ with both $3 < y - x$ and $x - y < 5$.
e. There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}^+$ with $x^n + y > 0$.

Problem 2: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "If $a, b \in \mathbb{Z}$ are both odd, then $a + b$ is even".

Let $a, b \in \mathbb{Z}$ be two arbitrary odd integers. Since a is odd, we can ________________. Since b is odd, we can ________________. Now notice that $a + b =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $a + b$ is an even integer. Since $a, b \in \mathbb{Z}$ were arbitrary, the result follows.

Problem 3: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "If $a \in \mathbb{Z}$ is even and $b \in \mathbb{Z}$, then ab is even".

Let $a, b \in \mathbb{Z}$ be arbitrary with a even. Since a is even, we can ________________. Now notice that $ab =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that ab is an even integer. Since $a, b \in \mathbb{Z}$ were arbitrary, the result follows.

Problem 4: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "For all $a \in \mathbb{Z}$, we have that $2a^3 + 6a - 3$ is odd".

Let $a \in \mathbb{Z}$ be arbitrary. We have $2a^3 + 6a - 3 =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $2a^3 + 6a - 3$ is odd. Since $a \in \mathbb{Z}$ was arbitrary, the result follows.

Problem 5: Consider the statement "If $a \in \mathbb{Z}$ is odd, then $4a + 1$ is even". Your friend claims to have a proof, and presents the following.

Let $a \in \mathbb{Z}$ an arbitrary odd integer. Since a is odd, we can fix $m \in \mathbb{Z}$ with $a = 2m + 1$. We then have that

$$\begin{aligned} 4a + 1 &= 4 \cdot (2m + 1) + 1 \\ &= 8m + 5 \\ &= 2 \cdot \left(\frac{8m + 5}{2} \right) \end{aligned}$$

We have shown the existence of an $n \in \mathbb{Z}$ with $4a + 1 = 2n$, so $4a + 1$ is even.

a. Pinpoint the error in your friend's argument. Be as specific as you can.
b. Is the statement in question true or false? Justify your answer carefully.