Problem Set 10: Due Friday, October 10

Problem 1: Consider the unique linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 2 & -5 \\ -6 & 15 \end{pmatrix}$$

Find, with explanation, vectors $\vec{u}, \vec{w} \in \mathbb{R}^2$ with $Null(T) = Span(\vec{u})$ and $\text{range}(T) = Span(\vec{w})$.

Problem 2: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Recall that

$$Null(T) = \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\}.$$

a. Show that if $\vec{v}_1, \vec{v}_2 \in Null(T)$, then $\vec{v}_1 + \vec{v}_2 \in Null(T)$.
b. Show that if $\vec{v} \in Null(T)$ and $c \in \mathbb{R}$, then $c \cdot \vec{v} \in Null(T)$.

Problem 3: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the unique linear transformation with

$$[T] = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}$$

Explain why T is invertible and calculate

$$T^{-1}\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right)$$

Problem 4: Consider the following system of equations:

$$\begin{array}{rcrcr} x & + & 4y & = & -3 \\ 2x & + & 5y & = & 8 \end{array}$$

a. Rewrite the above system in the form $A\vec{v} = \vec{b}$ for some matrix A and vector $\vec{b}$.
b. Explain why A is invertible and calculate A^{-1}.
c. Use A^{-1} to solve the system.

Problem 5: In this problem, let 0 denote the 2×2 zero matrix, i.e. the 2×2 matrix where all four entries are 0.
a. Give an example of a nonzero 2×2 matrix A with $A \cdot A = 0$.
b. Show that if A is invertible and $A \cdot A = 0$, then $A = 0$.
Note: Since 0 is not invertible, it follows from part b that there is no invertible matrix A with $A \cdot A = 0$.

Problem 6: Let A, B, C all be invertible 2×2 matrices. Must there exist a 2×2 matrix X with

$$A(X + B)C = I?$$

Either justify carefully or give a counterexample.