Written Assignment 8: Due Wednesday, November 27

Required Problems

Problem 1: In this problem, we determine which 2×2 matrices commute with *every* 2×2 matrix.
a. Show that if $r \in \mathbb{R}$ and we let

$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

then $AB = BA$ for every 2×2 matrix B.
b. Suppose that A is a 2×2 matrix which the property that $AB = BA$ for every 2×2 matrix B. Show that there exists $r \in \mathbb{R}$ such that

$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

Hint: For part b, make strategic choices for B to make your life as simple as possible. I suggest thinking about matrices with lots of zeros.

Interlude: Let A be an $m \times n$ matrix. Recall that we defined $Col(A)$ to be the span of the columns of A. Notice that with our definition of matrix multiplication, we can also write this as

$$Col(A) = \{A\vec{v} : \vec{v} \in \mathbb{R}^n\}$$

We defined the null space of a linear transformation, and we can similarly define the null space of a matrix as follows:

$$\mathcal{N}(A) = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \vec{0}\}$$

Now given an $n \times n$ matrix A, we say that A is *idempotent* if $A^2 = A$. For example, the zero matrix and the identity matrix are idempotent. More interesting examples are:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

One can check that the matrices in Challenge Problem 2 on Written Assignment 6 are all idempotent, as is any matrix representing the linear transformation in Challenge Problem 1 on Written Assignment 7. Intuitively, any matrix that represents a projection onto a subspace (like these examples do) will be idempotent, because projecting twice in succession gives the same result as just projecting once.

Problem 2: Let A be an $n \times n$ idempotent matrix.
a. Show that if A invertible, then $A = I_{n \times n}$.
b. Show that $\mathcal{N}(A) \cap Col(A) = \{\vec{0}\}$.
c. Show that $\mathcal{N}(A) + Col(A) = \mathbb{R}^n$. In other words, show that for every $\vec{v} \in \mathbb{R}^n$, there exists $\vec{u} \in \mathcal{N}(A)$ and $\vec{w} \in Col(A)$ with $\vec{v} = \vec{u} + \vec{w}$.

Challenge Problems

Problem 1: Let V and W be finite-dimensional vector spaces.
a. Suppose that $t_1 \colon V \to W$ is an injective linear transformation. Show that there exists a linear transformation $t_2 \colon W \to V$ such that $t_2 \circ t_1 = id_V$.
b. Suppose that $t_1 \colon V \to W$ is an surjective linear transformation. Show that there exists a linear transformation $t_2 \colon W \to V$ such that $t_1 \circ t_2 = id_W$.