Written Assignment 7: Due Wednesday, November 13

Required Problems

Problem 1: Define $t\colon \mathcal{P}_2 \to \mathbb{R}$ by letting

$$t(p(x)) = \int_0^1 p(x)\ dx$$

You know from Calculus that t is a linear transformation. In this problem we will bases for $\mathcal{R}(t)$ and $\mathcal{N}(t)$, and also compute $\text{rank}(t)$ and $\text{nullity}(t)$.
a. Give three examples, with justification, of elements in each of $\mathcal{R}(t)$ and $\mathcal{N}(t)$.
b. Guess potential bases for $\mathcal{R}(t)$ and $\mathcal{N}(t)$.
c. Prove that your choices in part b work.
d. What is $\text{rank}(t)$ and $\text{nullity}(t)$? Explain.

Problem 2: Let A, B, C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions (do not assume that A, B, C are vector spaces or that f and g are linear transformations). Show each of the following.
a. If $g \circ f$ is surjective, then g is surjective.
b. If $g \circ f$ is injective, then f is injective.
c. If $g \circ f$ is injective and f is surjective, then g is injective.
Note: Recall that to prove that f is injective, you should start by taking arbitrary $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$, and then try to deduce that $a_1 = a_2$. Also, to prove that g is surjective, you should start by taking an arbitrary $c \in C$, and show how to find $b \in B$ with $g(b) = c$.

Challenge Problems

Problem 1: Let V be a finite-dimensional vector space, and let U be a subspace of V. Prove that there exists a linear transformation $t\colon V \to U$ such that $t(\vec{u}) = \vec{u}$ for all $\vec{u} \in U$.