Written Assignment 5 : Due Wednesday, October 16

Problem 1: Let V be a vector space. On Written Assignment 4, you showed that if S and T are linearly independent subsets of V, then it may not be true that $S \cup T$ is linearly independent.
a. Suppose that S and T are both linearly independent subsets of V and also that $[S] \cap [T] = \{\vec{0}\}$. Show that $S \cup T$ is linearly independent.
b. Let U and W be two subspaces of $\mathbb{R}^7$ with $\dim(U) = \dim(W) = 4$. Show that $U \cap W \neq \{\vec{0}\}$.
Hint for a: Think about the proof of Theorem 2.III.1.10 in the book.

Problem 2: Recall from Written Assignment 4 that if V is a vector space and both U and W are subspaces of V, then we define

$$U + W = \{\vec{v} \in V : \text{There exists } \vec{u} \in U \text{ and } \vec{w} \in W \text{ with } \vec{v} = \vec{u} + \vec{w}\}$$

a. Show that if V is a finite-dimensional vector space and U and W are both subspaces of V, then

$$\dim(U + W) \leq \dim(U) + \dim(W).$$

b. Give a specific example of a finite-dimensional vector space V along with two subspaces U and W of V such that

$$\dim(U + W) < \dim(U) + \dim(W).$$

Hint for a: Start by fixing bases for U and W, and then try to build a set that spans $U + W$.