Written Assignment 3 : Due Wednesday, October 2

Note: When we use the word *or* in mathematics, we mean that at least one of the options happens (so this is the "inclusive" or). In other words, when we say "A or B", we are allowing the possibility that both A and B are true. Be sure to interpret *or* in this way in both Problem 1b and Problem 2b.

Problem 1: Let V be a vector space. Prove each of the following. For this problem, be very explicit and mention which of the vector space axioms you are using in each step of your argument.
a. Let $\vec{u}, \vec{v}, \vec{w} \in V$. Show that $\vec{u} + (\vec{v} + \vec{w}) = \vec{w} + (\vec{u} + \vec{v})$.
b. Suppose that $c \in \mathbb{R}$ and $\vec{v} \in V$ are such that $c \cdot \vec{v} = \vec{0}$. Show that either $c = 0$ or $\vec{v} = \vec{0}$.
Hint for b: In mathematics, one of the standard ways to prove a statement of the form "A or B" is to assume that A is false, and use this to prove that B must be true (or alternatively to assume that B is false, and use this to prove that A must be true). This allows you to use an additional assumption, which is extremely useful.

Problem 2: Let V be a vector space. Suppose that U and W are both subspaces of V.
a. Let $U \cap W$ be the intersection of U and W, i.e. $U \cap W = \{\vec{v} \in V : \vec{v} \in U \text{ and } \vec{v} \in W\}$. Show that $U \cap W$ is a subspace of V.
b. Let $U \cup W$ be the union of U and W, i.e. $U \cup W = \{\vec{v} \in V : \vec{v} \in U \text{ or } \vec{v} \in W\}$. By constructing an explicit example, show that $U \cup W$ need *not* be a subspace of V.
c. Let $U + W = \{\vec{v} \in V : \text{There exists } \vec{u} \in U \text{ and } \vec{w} \in W \text{ with } \vec{v} = \vec{u} + \vec{w}\}$. That is, $U + W$ is the set of all vectors in V that can be written as the sum of an element of U and an element of W. Show that $U + W$ is a subspace of V.