Homework 9: Due Monday, March 13

Problem 1: Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property holds, you should explain why. If a certain property fails, you should give a specific counterexample.
a. $X = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 3), (2, 3), (3, 1), (3, 2), (3, 3)\}$.
b. $X = \mathbb{Z}$ where $a \sim b$ means $a - b \neq 1$.
c. $X = \mathbb{Z}$ where $a \sim b$ means that both a and b are even.
d. $X = \mathbb{R}$ where $x \sim y$ means that $x^2 < y$.
e. $X = \mathcal{P}(\{1, 2, 3, 4, 5\})$ where $A \sim B$ means that $A \cap B \neq \emptyset$.
Recall: We sometimes use $a \sim b$ as shorthand for $(a, b) \in R$. Thus, for example, we can instead write b as saying that $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b \neq 1\}$.

Problem 2: Suppose that R and S are both relations on the same set X. Either prove or find a counterexample for each of the following:
a. If R and S are both symmetric, then $R \cup S$ is symmetric.
b. If R and S are both transitive, then $R \cup S$ is transitive.

Problem 3: Let R be a binary relation on a set X. Show that if R is transitive, then $R \circ R \subseteq R$.

Problem 4: For a reference example in this problem, consider the set $X = \{1, 2, 3, 4, 5\}$ and

$$R = \{(1, 1), (2, 2), (2, 3), (2, 5), (3, 2), (3, 3), (3, 5), (4, 4), (5, 2), (5, 3), (5, 5)\}.$$

It can be checked that R is an equivalence relation on X.
a. Recall that we code relations in ML as $(\alpha * \beta)$ sets. However, in the above example, we want to restrict the set R to be viewed as being a relation on $\{1, 2, 3, 4, 5\}$, not on all integers. To aid this, write an ML function obtainSet that takes an $(\alpha * \alpha)$ set R, assumed to be an equivalence relation on a subset of α, and produces that subset. For example, on the above equivalence relation, your program should produce the set [1, 2, 3, 4, 5].
b. Write an ML function thinSet that takes two inputs, an $(\alpha * \alpha)$ set that is assumed to be an equivalence relation on a subset X of α, and an α set Y that is assumed to be a subset of X, and produces a subset Z of Y with the property that every element of Y is related to exactly one element of Z. For example, thinSet applied to R and the set $[1, 2, 3]$ should produce either $[1, 2]$ or $[1, 3]$ (since 2 and 3 are related, we should omit one of them).
c. Write an ML function uniqueReps that takes as input an $(\alpha * \alpha)$ set, assumed to be an equivalence relation on a subset X of α, and produces a set of unique representatives of the equivalence classes. In other words, every element of X should be related to exactly one element of the set that you output. For example, on the above equivalence relation, your program should produce one of the following sets: [1, 2, 4], [1, 3, 4], or [1, 4, 5].
d. Write an ML function equivClasses that takes as input an $(\alpha * \alpha)$ set, assumed to be an equivalence relation on a subset X of α, and produces the set of equivalence classes. For example, on the above equivalence relation, your program should should [[1], [2, 3, 5], [4]], or some rearrangement thereof.