Homework 8: Due Wednesday, March 8

Problem 1: Recall that given $a, b \in \mathbb{Z}$, we defined $a \mid b$ to mean that there exists $m \in \mathbb{Z}$ with $b = am$.
a. Show that if $a, b, c \in \mathbb{Z}$, and both $a \mid b$ and $a \mid c$, then $a \mid b + c$.
b. Show that if $a \mid b$, then $a \mid bc$ for all $c \in \mathbb{Z}$.

Problem 2: Give a careful double containment proof that $\{3x^2 + 1 : x \in \mathbb{R}\} = \{x \in \mathbb{R} : x \geq 1\}$.

Problem 3: Give a careful double containment proof of each of the following:
a. For all sets A, B, and C, we have $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
b. If A and B are both subsets of a universal set $\mathcal{U}$, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Problem 4: Let R be a binary relation between X and Y.
a. Recall that if $a \in X$, then we defined

$$\mathcal{I}_R(a) = \{b \in Y : (a, b) \in R\}.$$

Write an ML function that computes $\mathcal{I}_R(a)$. That is, you should write an ML function image that takes two inputs, an element a of type α and an $(\alpha * \beta)$ set R, and produces the β set $\mathcal{I}_{\mathsf{R}}(\mathsf{a})$.
b. Recall that if $A \subseteq X$, then we defined

$$\mathcal{I}_R(A) = \{b \in Y : \text{There exists } a \in A \text{ with } (a, b) \in R\}.$$

Write an ML function that computes $\mathcal{I}_R(A)$. That is, you should write an ML function imageSet that takes two inputs, an α set cs and an $(\alpha * \beta)$ set R, and produces the β set $\mathcal{I}_{\mathsf{R}}(\mathsf{cs})$.
Note: I called the argument cs because as is a keyword in ML.

Problem 5:
a. Write an ML function that computes the inverse of a relation. That is, you should write an ML function inverseRelation that takes an input R of type $(\alpha * \beta)$ set, and produces the $(\beta * \alpha)$ set R^{-1}.
b. Write an ML function that computes the composition of two relations. That is, you should write an ML function composeRelations that takes two inputs, an $(\alpha * \beta)$ set R and $(\beta * \gamma)$ set S, and produces the $(\alpha * \gamma)$ set $\mathsf{S} \circ \mathsf{R}$.