Homework 12: Due Monday, April 17

Problem 1: On Homework 2, you wrote an ML function that computed

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n.$$

Prove that for all $a, r \in \mathbb{R}$ with $r \neq 1$, we have

$$a + ar + ar^2 + ar^3 + \cdots + ar^n = a \cdot \frac{r^{n+1} - 1}{r - 1}$$

for all $n \in \mathbb{N}$ by induction.

Problem 2: Define a sequence recursively by letting $a_0 = 42$ and letting

$$a_n = a_{n-1}^2 - 3a_{n-1} + 14$$

for all $n \in \mathbb{N}$ with $n \geq 1$. Show that $7 \mid a_n$ for all $n \in \mathbb{N}$.

Problem 3: Let $n \in \mathbb{N}$ with $n \geq 2$. Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1) \cdot n}$$

and prove it by induction.

Problem 4: In this problem, work with the type binTree as defined on Homework 11.
a. Write an ML function depth that takes as input an element t of type binTree, and produces the number of levels of t that contain a node. For example,

$$\mathsf{depth}(\mathsf{Node}(5, \mathsf{Node}(3, \mathsf{Null}, \mathsf{Null}), \mathsf{Node}(9, \mathsf{Null}, \mathsf{Null}))) = 2$$

since we have a level that contains 5, and a level that contains 3 and 9 (levels containing only Nulls are not counted).
b. Using your definition of depth in part a, and the definition of numNodes in Homework 11, use structural induction to show that

$$\mathsf{numNodes(t)} \leq 2^{\mathsf{depth}(t)} - 1$$

for all binary trees t.

Problem 5: On Homework 11, you turned trees into lists by writing flattenBinTree. In this problem, you will go in the other direction. That is, you will turn lists into (reasonably) balanced binary trees.
a. Write an ML function split that takes two inputs, a natural number k and a list bs, and returns a triple (cs, a, ds), where cs is the list consisting of the first k elements of bs, a is the next element of bs, and ds is the list consisting of the remaining elements of bs. For example,

$$\mathsf{split}(2, [0, 3, 4, 8, 10, 17])$$

should produce

$$([0, 3], 4, [8, 10, 17]).$$

You may assume that $0 \leq \mathsf{k} < \mathsf{length(bs)}$.

b. Write an ML function listToBinTree that takes as input a list of integers, and returns a balanced binary tree with the same elements. Moreover, your function should have the property that

$$\mathsf{flattenBinTree}(\mathsf{listToBinTree}(\mathsf{bs})) = \mathsf{bs}$$

for all lists of integers bs. If you need to make your tree slightly unbalanced, put an extra node on the left subtree instead of the right. For example, on input $[1, 2, 3, 4]$, your function should produce

$$\mathsf{Node}(3, \mathsf{Node}(2, \mathsf{Node}(1, \mathsf{Null}, \mathsf{Null}), \mathsf{Null}), \mathsf{Node}(4, \mathsf{Null}, \mathsf{Null})).$$

Hint: Part a will help you write part b. In each part, you should use recursion to solve a slightly easier instance of your problem, and use the answer to solve your instance. It might be helpful to use a let statement as in the sumProduct function on p. 73.