Homework 8: Due Monday, March 9

Note: For each of the counting problems, you must explain your solution. For example, if your answer is a product, describe the sequence of choices you are making and explain where each term comes from. Numerical answers without written justification will receive no credit.

Problem 1: Using the digits 1 through 9 only (so exclude 0), how many 13 digits numbers are there in which no two consecutive digits are the same?

Problem 2: Suppose that you are creating a password using 26 letters, 10 numbers, and 15 special characters. How many such 10-character passwords are possible if they must have exactly 6 letters, 2 numbers, and 2 special characters?

Problem 3: How many ways are there to pick two cards from a standard 52-card deck such that the first card is a spade and the second is not an ace? In this problem, order matters. So if you pick the 3 of spades followed by the 7 of spades, this is different from the 7 of spades followed by the 3 of spades.

Problem 4: Write a recursive Scheme function that takes as input a set as and a natural number $k \in \mathbb{N}$, and outputs the set of k-permutations of as. For example, on inputs as $=$ '(1 2 3) and $k = 2$, the function should return

'((1 2) (1 3) (2 1) (2 3) (3 1) (3 2))

(although possibly in a different order). If $k = 0$, your program should produce '(()) (because the empty sequence is technically a permutation of length 0). If k is greater than the number of elements in as, then your function should produce '() (because there are no such permutations). Write a paragraph explaining why your program works.

Problem 5: A *derangement* of the set $\{1, 2, 3, \dots, n\}$ is a permutation of $\{1, 2, 3, \dots, n\}$ such that the number i does not appear in position i for all $i \in \{1, 2, 3, \dots, n\}$. For example, $(2, 3, 1)$ is a derangement of $\{1, 2, 3\}$, but $(3, 2, 1)$ is not a derangement of $\{1, 2, 3\}$ (because the 2 is in position 2).

a. Write a Scheme function is-derangement? that takes as input a list as, assumed to be a permutation of the set $\{1, 2, 3, \dots,$ (length as)$\}$, and returns #t if as is a derangement, and #f otherwise. Write a paragraph explaining why your program works.

b. Using set-carve, write a Scheme function derangements that takes as input a natural number $n \in \mathbb{N}$, and returns the list of all derangements of $\{1, 2, 3, \dots, n\}$.

c. Write a Scheme function derangement-ratio that takes as input a natural number n, and returns the fraction (in decimal form) of permutations of $\{1, 2, 3, \dots, n\}$ that are derangements. For example, on input 3, your function should return .33333333... because 2 of the 6 permutations of $\{1, 2, 3\}$ are derangements, and $\frac{2}{6} = .33333333\dots$.

Cultural Aside: As you increase the input n to part c, the outputs appear to be approaching a certain number. It turns out that the outputs converge very rapidly to $\frac{1}{e}$.