Homework 6: Due Friday, February 27

Note: Throughout this assignment, let f_n be the the sequence of Fibonacci numbers, i.e. define $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{N}$ with $n \geq 2$.

Problem 1: Show that $f_n \leq 2^n$ for all $n \in \mathbb{N}$.

Problem 2: Show that

$$f_{n+1}f_{n-1} = f_n^2 + (-1)^n$$

for all $n \in \mathbb{N}^+$.

Problem 3: A friend tries to convince you that f_n if even for all $n \geq 3$. Here is their argument using strong induction. For the base case, notice that $f_3 = 2$, so f_3 is even. For the inductive step, suppose that the $n \geq 4$ and we know the result for all k with $3 \leq k < n$. Since $f_n = f_{n-1} + f_{n-2}$ and we know by induction that f_{n-1} and f_{n-2} are even, it follows that f_n is even (because the sum of two even numbers is even). Therefore, f_n is even for all $n \geq 3$.

Now you know that your friend's argument must be wrong because $f_7 = 13$ and 13 is not even. Pinpoint the fundamental error. Be as explicit and descriptive as you can.

Problem 4: Consider the following Scheme function:

```
(define fib
  (lambda (n curr next)
    (if (= n 0)
        curr
        (fib (- n 1) next (+ curr next)))))
```

Show that for all $k, n \in \mathbb{N}$, we have that (fib n $\mathsf{f_k}$ $\mathsf{f_{k+1}}$) equals f_{k+n}. Using the special case where $k = 0$, it follows that (fib n 0 1) equals f_n for all $n \in \mathbb{N}$.
Hint: Prove the statement "For all $k \in \mathbb{N}$, we have that (fib n $\mathsf{f_k}$ $\mathsf{f_{k+1}}$) equals f_{k+n}" by induction on n.

Interlude: We saw how fast the Euclidean Algorithm ran in class. However, it is not obvious why the algorithm terminates so quickly. In the next problem, we work to understand the theory behind the speed. Let $a, b \in \mathbb{N}$ with $b \neq 0$. Write $a = qb + r$ where $q, r \in \mathbb{N}$ and $0 \leq r < b$. Notice that after one step of the algorithm, the new second argument r may not be much smaller than the original second argument b. For example, if $a = 77$ and $b = 26$, then we have $q = 2$ and $r = 25$. However, it turns out that after *two* steps of the Euclidean Algorithm, the new second argument will be at most half the size of the original second argument. This is what we will prove.

Problem 5: Let $a, b \in \mathbb{N}$ with $b \neq 0$. In the first step of the algorithm, we write $a = qb + r$ where $q, r \in \mathbb{N}$ and $0 < r < b$ (we can assume that $r \neq 0$ because otherwise the algorithm stops at the next step). In the next step of the algorithm, we write $b = pr + s$ where $0 \leq s < r$. Show that $s < \frac{b}{2}$. Thus, the second argument is at least cut in half after two steps of the Euclidean algorithm, which is just like our functions power-mod-fast and multiply-fast.
Hint: Break the problem into cases based on how large r is.