Homework 10: Due Monday, April 6

Problem 1: Let $n \in \mathbb{N}^+$ and let $x \in \mathbb{R}$ with $x \geq 0$. Use the Binomial Theorem to show that $(1+x)^n \geq 1+nx$.

Problem 2: Determine (with explanation), the value of each of the following sums:
a. Given $n \in \mathbb{N}$:

$$\sum_{k=0}^{n} 2^k \cdot \binom{n}{k} = \binom{n}{0} + 2 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + 8 \cdot \binom{n}{3} + \cdots + 2^n \cdot \binom{n}{n}.$$

b. Given $n \in \mathbb{N}$ with $n \geq 2$:

$$\sum_{k=1}^{n} (-1)^{k-1} \cdot k \cdot \binom{n}{k} = \binom{n}{1} - 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} - 4 \cdot \binom{n}{4} + \cdots + (-1)^{n-1} \cdot n \cdot \binom{n}{n}.$$

Hint: Start with the Binomial Theorem, and plug in strategic choices for x and y. Some Calculus may be helpful.

Problem 3: For all $k, n \in \mathbb{N}^+$ with $k \leq n$, we know that $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$ since each side counts the number of ways of selecting a committee consisting of k people, including a distinguished president of the committee, from a group of n people.
a. Let $k, m, n \in \mathbb{N}^+$ with $m \leq k \leq n$. Give a combinatorial proof (i.e. argue that both sides count the same set) of the following:

$$\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}$$

This generalizes the above result (which is the special case where $m = 1$).
b. Let $m, n \in \mathbb{N}^+$ with $m \leq n$. Find a simple formula for:

$$\sum_{k=m}^{n} \binom{n}{k} \cdot \binom{k}{m}$$

and explain why it is true.

Interlude: Recall that we defined (integer labeled) binary trees recursively as follows:

- null is a binary tree.
- If a is an integer and t1 and t2 are binary trees, then (list a t1 t2) is a binary tree.

Problem 4: A *leaf* of a binary tree is a node without children. Thus, in our coding of binary trees, a leaf is node where both the corresponding t1 and t2 are null. Write a Scheme program num-leaves-bin-tree that takes a binary tree as input, returns the number of leaves in the tree.

Problem 5: Write a Scheme program flatten-bin-tree that takes a binary tree as input, and returns the list of all integers in order when read across the tree from left to right. For example, on the input

(list 1 (list 4 null null) (list 2 (list 3 null null) null)),

the program should output

'(4 1 3 2).

On the input

(list 7 (list 4 (list 2 (list 1 null null) (list 3 null null)) (list 5 null null)) (list 11 null null)),

(which is our tree from class on Monday, March 30), the program should output

′(1 2 3 4 5 7 11).