Homework 1: Due Wednesday, January 28

Problem 1: The *natural numbers* are the numbers $0, 1, 2, 3, \ldots$. Write a Scheme function multiply that takes two inputs a and b, assumed to be natural numbers, and produces the product $\mathsf{a} \cdot \mathsf{b}$ as output. Your function should be a basic recursive function that uses only the following operations:

- Addition of natural numbers.
- Subtracting 1 from a (nonzero) natural number.

Also, write a paragraph explaining why your program works. What mathematical properties of multiplication and addition are you using?

Problem 2: Write a tail-recursive solution to Problem 1. Explain why your program works.

Problem 3: The *integers* are the numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. Write a version of your basic recursive program in Problem 1 that works more generally if a and b are integers. Your function should be a basic recursive function that uses only the following operations:

- Addition of integers.
- Subtraction of integers.

Explain why your program works.

Problem 4: Given integers a and b, how many recursive calls does your program in Problem 3 make when computing (multiply a b)? Explain.

Problem 5: The *rational numbers* are the numbers that we can write as a quotient $\frac{a}{b}$, where a and b are integers and $b \neq 0$. In other words, the rational numbers are those that can be expressed as fractions. For example, $\frac{1}{2}$ and $\frac{-7}{11}$ are rational numbers. Every integer is a rational number, since can view 3 as $\frac{3}{1}$, for example. Notice that some rational numbers have multiple representations, such as $\frac{1}{2} = \frac{2}{4}$. Of course, if we know how to add/subtract/multiply integers, then we can add/subtract/multiply rational numbers:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}, \qquad \text{and} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Assume that we have developed a representation of rational numbers in Scheme (such as representing a rational q as a certain special ordered pair (a b) of integers). Suppose also that we have developed Scheme functions (add q r) and (subtract q r) that produce the sum/difference of two rational numbers q and r. Now, if we "break up" rationals into numerator and denominator, then we can use the above formula for multiplication of rationals together with your program in Problem 3 to write a program to multiply rational numbers. However, suppose that you try to define multiplication of rational numbers in terms of addition and subtraction of rational numbers, without breaking them up into numerators and denominators. In other words, suppose that you want to define (multiply q r) recursively in terms of add and subtract like in Problem 3. What obstacles do you face in trying to make this work? Explain in as much detail as possible.