Mathematical Statements

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1 Mathematical Statements and Mathematical Truth

Unfortunately, many people view mathematics only as complicated equations and elaborate computational techniques (or algorithms) that lead to the correct answers to a narrow class of problems. Although each of these are indeed aspects of mathematics, neither reflects the true nature of the subject. Mathematics, at its core, is about determining *truth*, at least for certain precise mathematical statements. Before we consider some examples, let's recall some notation and terminology for the standard "universes" of numbers.

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of *natural numbers*.
- $\mathbb{N}^+ = \{1, 2, 3, \dots\}$ is the set of positive natural numbers.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of *integers*.
- $\mathbb{Q}$ is the set of *rational* numbers, i.e. those numbers that can be written as a fraction (i.e. quotient) of integers. For example, $\frac{1}{2}$, $\frac{-3}{17}$, etc. are all rational numbers. Notice that all integers are rational numbers because we can view 5 as $\frac{5}{1}$, for example.
- $\mathbb{R}$ is the set of *real* numbers, i.e. those numbers that we can express as a possibly infinite decimal. Every rational number is a real number, but π, e, $\sqrt{2}$, etc. are all real numbers that are not rational.

There are other important universes of numbers, such as the complex numbers $\mathbb{C}$ and others that will be encountered in Abstract Algebra. However, we will focus on the above examples in our study. To denote that a given number n belongs to one of the above collections, we will use the $\in$ symbol. For example, we can write $n \in \mathbb{Z}$ as shorthand for "n is an integer". We will elaborate on how to use the symbol $\in$ when we discuss more general set theory notation.

Returning to our discussion of truth, a mathematical statement is either objectively true or false, without reference to the outside world and without any additional conditions or information. For some examples, consider the following (we've highlighted some key words that we will discuss in the next few sections):

1. $35 + 81$ is equal to 116.
2. The sum of two odd integers is **always** an even integer.
3. The difference of two prime numbers is **always** an even integer.
4. **There exists** a simultaneous solution to the three equations

$$\begin{array}{rcrcrcr} 2x & & & + & 8z & = & 6 \\ 7x & - & 3y & + & 18z & = & 15 \\ -3x & + & 3y & - & 2z & = & -1 \end{array}$$

   in $\mathbb{R}^3$, i.e. **there exists** a choice of real numbers for x, y, and z making all three equations true.

5. The remainder when dividing 333^{2856} by 2857 is 1.
6. **Every** continuous function is differentiable.
7. **Every** differentiable function is continuous.
8. **There exist** positive natural numbers a, b, c with $a^3 + b^3 = c^3$.
9. The digits of π eventually form a repeating sequence.
10. The values of $0, 1, 2, \ldots, 9$ occur equally often in the infinite decimal expansion of π.

Which of these 10 assertions are true and which are false? In many cases, the answer is not obvious. Here are the results:

1. True. This statement can be verified by a simple calculation.
2. True. However, it's not immediately obvious how we could ever verify it. After all, there are infinitely many odd numbers, so we can't simply try them all.
3. False. To show that it is false, it suffices to give just one counterexample. Notice that 7 and 2 are prime, but $7 - 2 = 5$ and 5 is not even.
4. False. Again, it may not be obvious how to show that *no* possible choice of x, y, and z exist. We will develop systemic ways to solve such problems later.
5. True. It is possible to verify this by calculation (by using a suitably programmed computer). However, there are better ways to understand that this is true, as you will see in Elementary Number Theory or Abstract Algebra.
6. False. The function $f(x) = |x|$ is continuous everywhere but is not differentiable at 0.
7. True. See Calculus or Analysis.
8. False. This is a special case of something called Fermat's Last Theorem, and it is quite difficult to show (see Algebraic Number Theory).
9. False. This follows from the fact that π is an irrational number, i.e. not an element of $\mathbb{Q}$, but this is not easy to show.
10. We still don't know whether this is true or false! Numerical evidence (checking the first billion digits directly, for example) suggests that it may be true. Mathematicians have thought about this problem for a century, but we still do not know how to answer it definitively.

Recall that a mathematical statement must be either true or false. In contrast, an equation is typically neither true nor false when viewed in isolation, and hence is not a mathematical statement. For example, it makes no sense to ask if $y = 2x + 3$ is true or false, because it depends which numbers we plug in for x and y. When $x = 6$ and $y = 15$, then the statement becomes true, but when $x = 3$ and $y = 7$, the statement is false. For a more interesting example, the equation

$$(x + y)^2 = x^2 + 2xy + y^2$$

is not a mathematical statement as given, because we have not been told how to interpret the x and the y. Is the statement true when x is my cat Cayley and y is my cat Maschke? (Adding them together is scary enough, and I don't even want to think about what it would mean to *square* them.) We need to provide context for where x and y can come from. To fix this, we can write

"**For all** real numbers x and y, we have $(x+y)^2 = x^2 + 2xy + y^2$",

which is now a true mathematical statement.

You may think that the statement $(x+y)^2 = x^2 + y^2$ is false, but again it is not a valid mathematical statement as written. We can instead say that the statement

"**For all** real numbers x and y, we have $(x+y)^2 = x^2 + y^2$"

is false because $(1+1)^2 = 4$ while $1^2 + 1^2 = 2$. However, the mathematical statement

"**There exist** real numbers x and y such that $(x+y)^2 = x^2 + y^2$"

is true because $(1+0)^2$ does equal $1^2 + 0^2$.

Here are a few other examples of statements that are *not* mathematical statements.

- $F = ma$ and $E = mc^2$: Our current theories of physics say that these equations are true in the real world whenever the symbols are interpreted properly, but mathematics on its own is a different beast. As written, these equations are neither true nor false from a mathematical perspective. For example, if $F = 4$, $m = 1$, and $a = 1$, then $F = ma$ is certainly false.

- $a^2 + b^2 = c^2$: Unfortunately, most people "remember" this as the Pythagorean Theorem. However, it is not even a mathematical statement as written. We could fix it by writing "**For all** right triangles with side lengths a, b, and c, where c is the length the hypotenuse, we have that $a^2 + b^2 = c^2$", in which case we have a true mathematical statement.

- Talking Heads is the greatest band of all time: Of course, different people can have different opinions about this. I may believe that the statement is true, but the notion of "truth" here is very different from the objective notion of truth necessary for a mathematical statement.

- Shakespeare wrote *Hamlet*: This is almost certainly true, but it's not a mathematical statement. First, it references the outside world. Also, it's at least conceivable that with new evidence, we might change our minds. For example, perhaps we'll learn that Shakespeare stole the work of somebody else.

In many subjects, a primary goal is to determine whether certain statements are true or false. However, the methods for determining truth vary between disciplines. In the natural sciences, truth is often gauged by appealing to observations and experiments, and then building a logical structure (perhaps using some mathematics) to convince others of a claim. Economics arguments are built through a combination of current and historical data, mathematical modeling, and rhetoric. In both of these examples, truth is always subject to revision based on new evidence. In contrast, mathematics has a unique way of determining the truth or falsity of a given statement: we provide an airtight, logical *proof* that verifies its truth with certainty. Once we've succeeded in finding a correct proof of a mathematical statement, we know that it must be true for eternity. Unlike the natural sciences, we do not have tentative theories that are extremely well-supported but may be overthrown with new evidence. Thus, mathematics does not have the same types of revolutions like plate tectonics, evolution by natural selection, the oxygen theory of combustion (in place of phlogiston), relativity, quantum mechanics, etc. which overthrow the fundamental structure of a subject and cause a fundamental shift if what statements are understood to be true.

To many, the fact that mathematicians require a complete logical proof with absolute certainty seems strange. Doesn't it suffice to simply check the truth of statement in many instances, and then generalize it to a universal law? Consider the following example. One of the true statements mentioned above is that there are no positive natural numbers a, b, c with $a^3 + b^3 = c^3$, so we can not obtain a cube by adding 2 cubes.

The mathematician Leonhard Euler conjectured that a similar statement held for fourth powers, i.e. that we can not obtain a fourth power by adding 3 fourth powers. More formally, he conjectured that there are no positive natural numbers a, b, c, d with $a^4 + b^4 + c^4 = d^4$. For over 200 years it seemed reasonable to believe this might be true, as it held for all small examples and was a natural generalization of a true statement. However, it was eventually shown that there indeed are examples where the sum of 3 fourth powers equals a fourth power, such as the following:

$$95800^4 + 217519^4 + 414560^4 = 422481^4.$$

In fact, this example is the smallest one possible. Thus, even though $a^4 + b^4 + c^4 \neq d^4$ for all values positive natural numbers $a, b, c,$ and d having at most 5 digits, the statement does not hold generally.

In spite of this example, you may question the necessity of proofs for mathematics relevant to the sciences and applications, where approximations and occasional errors or exceptions may not matter so much. There are many historical reasons why mathematicians have embraced complete, careful, and logical proofs as **the** way to determine truth in mathematics independently from applications. In later math classes, you may explore some of these internal historical aspects, but here are three direct reasons for this approach:

- Mathematics should exist independently from the sciences because sometimes the same mathematics applies to different subjects. It is possible that edge cases which do not matter in one subject (say economics or computer science) might matter in another (like physics). The math needs to be consistent and coherent on its own without reference to the application.
- In contrast to the sciences where two generally accepted theories that contradict each other in some instance can coexist for long periods of time (such as relativity and quantum mechanics), mathematics can not sustain such inconsistencies. As we'll see, one reason for this is that mathematics allows a certain type of argument called proof by contradiction. Any inconsistency at all would allow us to draw all sorts of erroneous conclusions, and the logical structure of mathematics would unravel.
- Unlike the sciences, many areas of math are not subject to direct validation through a physical test. An idea in physics or chemistry, arising from either a theoretical predication or a hunch, can be verified by running an experiment. However, in mathematics we often have no way to reliably verify our guesses through such means. As a result, proofs in mathematics can be viewed as the analogues of experiments in the sciences. In other words, since mathematics exists independently from the sciences, we need an internal check for our intuitions and hunches, and proofs play this role.

In the previous examples of mathematical statements , we highlighted two key phrases that appear incredibly often in mathematical statements: **for all** and **there exists**. These two phrases are called *quantifiers* in mathematics, and they form the building blocks of more complicated expressions. Occasionally, these quantifiers appear disguised by a different word choice. Here are a few phrases that mean precisely the same thing in mathematics:

- **For all**: For every, For any, Every, Always, ...
- **There exists**: There is, For some, We can find, ...

These phrases mean what you might expect. For example, saying that a statement of the form "For all a, ..." is true means that whenever we plug in any particular value for a into the ... part, the resulting statement is true. Similarly, saying that a statement of the form "There exists a, ..." is true means that there is at least one (but possibly more) choice of a value to plug in for a so that the resulting statement is true. Notice that we are *not* saying that there is exactly one choice. Also, be careful in that the phrase "for some" used in everyday conversation could be construed to mean that there need to be several (i.e. more than one) values to plug in for a to make the result true, but in math it is completely synonymous with "there exists".

So how do we prove that a statement that starts with "there exists" is true? For example, consider the following statement:

$$\text{"There exists } a \in \mathbb{Z} \text{ such that } 2a^2 - 1 = 71\text{".}$$

From your training in mathematics up to this point, you may see the equation at the end and immediately rush to manipulate it using the procedures that you've been taught for years. Before jumping into that, let's examine the logical structure here. In order to convince somebody that the statement is true, we need only find (at least) one particular value to plug in for a so that when we compute $2a^2 - 1$ we obtain 71. Right? In other words, if all that we care about is knowing for sure that the statement is true, we just need to verify that some $a \in \mathbb{Z}$ has this property. Suppose that we happen to stumble across the number 6 and notice that

$$\begin{aligned} 2 \cdot 6^2 - 1 &= 2 \cdot 36 - 1 \\ &= 72 - 1 \\ &= 71. \end{aligned}$$

At this point, we can assert with confidence that the statement is true and in fact what we've just carried out is a complete proof. Now you may ask yourself "How did we know to plug in 6 here?", and that is a good question. However, there is a difference between the creative leap we took and the routine verification that it worked. Perhaps we arrived at 6 by plugging in numbers until we got lucky. Perhaps we sacrificed a chicken to get the answer. Perhaps we had a vision. Maybe you copied the answer from a friend or from online (note: don't do this). Now we do care very much about the underlying methods to find a, both for ethical reasons and because sacrificing a chicken may not work if we change the equation slightly. However, for the logical purposes of this argument, the way that we arrived at our value for a does not matter.

We're (hopefully) all convinced that we have verified that the statement "There exists $a \in \mathbb{Z}$ such that $2a^2 - 1 = 71$" is true, but as mentioned we would like to have routine methods to solve similar problems in the future so that we do not have to stumble around in the dark nor invest in chicken farms. Of course, the tools to do this are precisely the material that you learned years ago in elementary algebra. One approach is to perform operations on both sides of the equality with the goal of isolating the a. If we add 1 to both sides, we arrive at $2a^2 = 72$, and after dividing by sides by 2 we conclude that $a^2 = 36$. At this point, we realize that there are two solutions, namely 6 and -6. Alternatively, we can try bringing the 71 over and factoring. By the way, this method found two solutions, and indeed -6 would have worked above. However, remember that proving a "there exists" statement means just finding at least one value that works, so it didn't matter that there was more than one solution.

Let's consider the following more interesting example of a mathematical statement:

$$\text{"There exists } a \in \mathbb{R} \text{ such that } 2a^5 + 2a^3 - 6a^2 + 1 = 0\text{".}$$

It's certainly possible that we might get lucky and find a real number to plug in that verifies the truth of this statement. But if the chicken sacrificing doesn't work, you may be stymied about how to proceed. However, if you remember Calculus, then there is a nice way to argue that this statement is true without actually finding a particular value of a. The key fact is the Intermediate Value Theorem from Calculus, which says that if $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous function that is positive at some point and negative at another, then it must be 0 at some point as well. Letting $f(x) = 2x^5 + 2x^3 - 6x^2 + 1$, we know from Calculus that $f(x)$ is continuous. Since $f(0) = 1$ and $f(1) = -1$, it follows from the Intermediate Value Theorem that there is an $a \in \mathbb{R}$ (in fact between 0 and 1) such that $f(a) = 0$. Thus, we've proven that the above statement is true, so long as you accept the Intermediate Value Theorem. Notice again that we've established the statement without actually exhibiting an a that works.

We can make the above question harder by performing the following small change to the statement:

$$\text{"There exists } a \in \mathbb{Q} \text{ such that } 2a^5 + 2a^3 - 6a^2 + 1 = 0\text{".}$$

Since we do not know what the value of a that worked above was, we are not sure whether it is an element of $\mathbb{Q}$. In fact, questions like this are a bit harder. There is indeed a method to determine the truth of a statement like this, but that's for another course (see Abstract Algebra). The takeaway lesson here is that mathematical statements that look quite similar might require very different methods to solve.

How do we prove that a statement that starts with "for all" is true? For example, consider the following statement:

$$\text{“For all } a, b \in \mathbb{R}, \text{ we have } (a+b)^2 = a^2 + 2ab + b^2\text{”.}$$

In the previous section, we briefly mentioned this statement, but wrote it slightly differently as:

$$\text{“For all real numbers } x \text{ and } y\text{, we have } (x+y)^2 = x^2 + 2xy + y^2\text{”.}$$

Notice that these are both expressing the exact same thing. We only replaced the phrase "real numbers" by the symbol $\mathbb{R}$ and changed our choice of letters. Since the letters are just placeholders for the "for all" quantifier, these two mean precisely the same thing. Ok, so how do we prove the first statement? The problem is that there are infinitely many elements of $\mathbb{R}$ (so infinitely many choices for *each* of a and b), and hence there is no possible way to examine each possible pair in turn and ever hope to finish.

The way around this obstacle is write a general argument that works regardless of the values for a and b. In other words, we're going to take two completely *arbitrary* elements of $\mathbb{R}$ that we will name as a and b so that we can refer to them, and then argue that the result of computing $(a+b)^2$ is the same thing as the result of computing $a^2 + 2ab + b^2$. By taking arbitrary elements of $\mathbb{R}$, our argument will work no matter which particular numbers are actually chosen for a and b. Thus, the way to handle infinitely many choices is to give an argument that works no matter which of the infinitely many choices is taken for a and b.

Now in order to do this, we have to start somewhere. After all, with no assumptions at all about how $+$ and $\cdot$ work, or what squaring means, we have no way to proceed. Ultimately, mathematics starts with basic axioms explaining how certain fundamental mathematical objects and operations work, and builds up everything from there. We won't go into all of those axioms here, but for the purposes of this discussion we will make use of the following fundamental facts about the real numbers:

- Commutative Law (for multiplication): For all $x, y \in \mathbb{R}$, we have $x \cdot y = y \cdot x$.
- Distributive Law: For all $x, y, z \in \mathbb{R}$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$.

These facts are often taken as two (of about 12) of the axioms for the real numbers. It is also possible to prove them from a construction of the real numbers (see Analysis) using more fundamental axioms. In any event, we can use them to prove the above statement follows. Let $a, b \in \mathbb{R}$ be arbitrary. We then have that $a + b \in \mathbb{R}$, and

$$\begin{aligned}
(a+b)^2 &= (a+b)\cdot(a+b) && \text{(by definition)}\\
&= (a+b)\cdot a + (a+b)\cdot b && \text{(by the Distributive Law)}\\
&= a\cdot(a+b) + b\cdot(a+b) && \text{(by the Commutative Law)}\\
&= a\cdot a + a\cdot b + b\cdot a + b\cdot b && \text{(by the Distributive Law)}\\
&= a^2 + a\cdot b + a\cdot b + b^2 && \text{(by the Commutative Law)}\\
&= a^2 + 2ab + b^2. && \text{(by definition)}
\end{aligned}$$

Focus on the logic, and not the algebraic manipulations. First, you should view this chain of equalities by reading it in order. We are claiming that $(a+b)^2$ equals $(a+b)\cdot(a+b)$ in the first line. Then the second line says that $(a+b)\cdot(a+b)$ equals $(a+b)\cdot a + (a+b)\cdot b$ by the Distributive Law. Following this is an assertion that the third and fourth expressions are equal by the Commutative Law, etc. In the second line, notice that $a+b$ is in particular some real number (call it x), and then by viewing $a+b$ also as the sum of two real numbers (playing the role of y and z), we can apply the Distributive Law. If you believe all of the steps, then we have shown that for our completely arbitrary choice of a and b in $\mathbb{R}$, the first and second expressions are equal, the second and third expressions are equal, the third and fourth expressions are equal, etc. Since equality is transitive (i.e. if $x = y$ and $y = z$, then $x = z$), we conclude that $(a+b)^2 = a^2 + 2ab + b^2$. We have taken completely arbitrary $a, b \in \mathbb{R}$, and verified the statement in question, so we can now assert that the "For all" statement is true.

As a quick aside, now that we know that $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in \mathbb{R}$, we can use this fact whenever we have two real numbers. We can even conclude that the statement

$$\text{“For all } a, b \in \mathbb{R}, \text{ we have } (2a+3b)^2 = (2a)^2 + 2(2a)(3b) + (3b)^2\text{”}$$

is true. How does this follow? Consider completely arbitrary $a, b \in \mathbb{R}$. We then have that $2a \in \mathbb{R}$ and $3b \in \mathbb{R}$, and thus we can *apply* our previous result to the two numbers $2a$ and $3b$. We are *not* setting "$a = 2a$" or "$b = 3b$" because it does not make sense to say that $a = 2a$ if a is anything other than 0. We are simply using the fact that if a and b are real numbers, then $2a$ and $3b$ are also real numbers, so we can insert them in for the placeholder values of a and b in our result. Always think of the (arbitrary) choice of letters used in "there exists" and "for all" statements as empty vessels that could be filled with any appropriate value.

We've discussed the basic idea behind proving that a "there exists" or a "for all" statement is true. How do we we prove that such statements are false? The cheap answer is to prove that its *negation* is true! In other words, if we want to prove that

$$\text{“There exists } a \text{ such that } \dots\text{”}$$

is false, we can instead prove that

$$\text{“}\textbf{Not} \text{ (There exists } a \text{ such that } \dots)\text{”}$$

is true. This sounds great, but now we have this **Not** in the front, so the statement as a whole is no longer a "there exists" statement. However, to show that there does not exist an a with a certain property, we need to show that every a *fails* to have that property. Thus, we can instead show that the statement

$$\text{“For all } a, \text{we have } \textbf{Not} \ (\dots)\text{”}$$

is true. For example, suppose that we want to show that the statement

$$\text{“There exists } a \in \mathbb{R} \text{ such that } a^2 + 2a = -5\text{”}$$

is false. By the above discussion, we can instead show that

$$\text{“}\textbf{Not} \text{ (There exists } a \in \mathbb{R} \text{ such that } a^2 + 2a = -5)\text{”}$$

is true, which is the same as showing that

$$\text{“For all } a \in \mathbb{R}, \text{we have } \textbf{Not}(a^2 + 2a = -5)\text{”}$$

is true, which is the same as showing that

$$\text{“For all } a \in \mathbb{R}, \text{we have } a^2 + 2a \neq -5\text{”}$$

is true. In other words, we can move the negation past the "there exists" as long as we change it to a "for all" when doing so. How can we show that this last statement is true? Consider an arbitrary $a \in \mathbb{R}$. Notice that

$$\begin{aligned} a^2 + 2a &= (a^2 + 2a + 1) - 1 \\ &= (a+1)^2 - 1 \\ &\geq 0 - 1 && \text{(because squares of reals are nonnegative)} \\ &= -1. \end{aligned}$$

We have shown that given any arbitrary $a \in \mathbb{R}$, we have $a^2 + 2a \geq -1$, and hence $a^2 + 2a \neq -5$. We conclude that the statement

$$\text{"For all } a \in \mathbb{R}, \text{we have } a^2 + 2a \neq -5\text{"}$$

is true, and hence the statement

$$\text{"There exists } a \in \mathbb{R} \text{ with } a^2 + 2a = -5\text{"}$$

is false. Can you see a way to solve this problem using Calculus?

Similarly, if we want to prove that

$$\text{"For all } a, \text{ we have } \dots\text{"}$$

is false, then we can instead show that

$$\text{"}\textbf{Not} \text{ (For all } a, \text{ we have } \dots)\text{"}$$

is true, which is the same as showing that

$$\text{"There exists } a \text{ such that } \textbf{Not} \ (\dots)\text{"}$$

is true. In general, we can move a **Not** past one of our two quantifiers at the expense of *flipping* the quantifier to the other type.

Life becomes more complicated when a mathematical statement involves both types of quantifiers in an alternating fashion. For example, consider the following two statements:

1. "For all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $m < n$".
2. "There exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, we have $m < n$".

At first glance, these two statements appear to be essentially the same. After all, they both have "for all $m \in \mathbb{N}$", both have "there exists $n \in \mathbb{N}$", and both end with the expression "$m < n$". Does the fact that these quantifiers appear in different order matter?

Let's examine the first statement more closely. Notice that it has the form "For all $m \in \mathbb{N} \dots$". In order for this statement to be true, we want to know whether we obtain a true statement *whenever* we plug in a particular natural number for m in the "..." part. In other words, we're asking if *all* of the infinitely many statements:

- "There exists $n \in \mathbb{Z}$ such that $0 < n$".
- "There exists $n \in \mathbb{Z}$ such that $1 < n$".
- "There exists $n \in \mathbb{Z}$ such that $2 < n$".
- "There exists $n \in \mathbb{Z}$ such that $3 < n$".

- ...

are true. Looking through each of these, it does indeed appear that they are all true: We can use $n = 1$ in the first one, then $n = 2$ in the second, etc. However, there are infinitely many statements, so we can't actually check each one in turn and hope to finish. We need a general argument that works no matter which value m takes. Now given any *arbitrary* $m \in \mathbb{N}$, we can verify that by taking $n = m + 1$, we obtain a true statement. Here is how we would write this argument up formally.

Proposition 1.1. *For all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $m < n$.*

Proof. Let $m \in \mathbb{N}$ be arbitrary. We then have that $m+1 \in \mathbb{N}$ and $m < m+1$, so we have shown the existence of an $n \in \mathbb{N}$ with $m < n$ (namely $m + 1$). Since $m \in \mathbb{N}$ was arbitrary, the result follows. □

Let's pause to note a few things about this argument. First, we've labeled the statement as a proposition. By doing so, we are making a claim that the statement to follow is a true statement, and that we will be providing a proof. Alternatively, we sometimes will label a statement as a "theorem" instead of a "proposition" if we want to elevate it to a position of prominence (typically theorems say something powerful, surprising, or incredibly useful). In the proof, we are trying to argue that a "for all" statement is true, so we start by taking an *arbitrary* element of $\mathbb{N}$. Although this m is arbitrary, it is *not* varying. Instead, once we take an arbitrary m, it is now one fixed number that we can use in the rest of the argument. For this particular but arbitrary $m \in \mathbb{N}$, we now want to argue that a certain "there exists" statement is true. In order to do this, we need to exhibit an example of an n that works, and verify it for the reader. Since we have a fixed $m \in \mathbb{N}$ in hand, the n that we pick can depend on that m. In this case, we simply verify that $m + 1$ works as a value for n. As in the examples given above, we do not need to explain why we chose to use $m + 1$, only that the resulting statement is true. In fact, we could have chose $m + 2$, or $5m + 3$, etc. In the last line, we point out that since $m \in \mathbb{N}$ was arbitrary, and we succeeded in verifying the part inside the "for all" for this m, we can assert that the "for all" statement is true. Finally, the square box at the end of the argument indicates that the proof is over, and so the next paragraph (i.e. this one) is outside the scope of the argument.

Let's move on to the second of our two statements above. Notice that it has the form "There exists $n \in \mathbb{N} \dots$". In order for this statement to be true, we want to know whether we can find *one* value for n such that we obtain a true statement in the "..." part after plugging it in. In other words, we're asking if *any* of the infinitely many statements

- "For all $m \in \mathbb{N}$, we have $m < 0$".
- "For all $m \in \mathbb{N}$, we have $m < 1$".
- "For all $m \in \mathbb{N}$, we have $m < 2$".
- "For all $m \in \mathbb{N}$, we have $m < 3$".
- ...

is true. Looking through each of these, it appears that every single one of them is false, i.e. *none* of them are true. Thus, it appears that the second statement is false. We can formally prove that it is false by proving that its negation is true. Applying our established rules for how to negate across quantifiers, to show that

$$\text{"}\textbf{Not} \text{ (There exists } n \in \mathbb{N} \text{ such that for all } m \in \mathbb{N}, \text{ we have } m < n)\text{"}$$

is true, we can instead show that

$$\text{"For all } n \in \mathbb{N}, \ \textbf{Not} \text{ (for all } m \in \mathbb{N}, \text{ we have } m < n)\text{"}$$

is true, which is same as showing that

$$\text{“For all } n \in \mathbb{N}, \text{ there exists } m \in \mathbb{N} \text{ with } \textbf{Not}(m < n)\text{”}$$

is true, which is the same as showing that

$$\text{“For all } n \in \mathbb{N}, \text{ there exists } m \in \mathbb{N} \text{ with } m \geq n\text{”}.$$

is true. We now prove that this final statement is true, which is the same as showing that our original second statement is false.

Proposition 1.2. *For all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $m \geq n$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary. Notice that $n \geq n$ is true, so we have shown the existence of an $m \in \mathbb{N}$ with $m \geq n$. Since $n \in \mathbb{N}$ was arbitrary, the result follows. □

In fact, if we think about it for a moment, we did not have to write a new formal proof here. We wanted to prove that

$$\text{“For all } n \in \mathbb{N}, \text{ there exists } m \in \mathbb{N} \text{ with } m \geq n\text{”}$$

is true. In Proposition 1.1, we showed that

$$\text{“For all } m \in \mathbb{N}, \text{ there exists } n \in \mathbb{N} \text{ with } m < n\text{”}$$

is true. Now remember that the letters are simply placeholders, so we can restate Proposition 1.1 as

$$\text{“For all } n \in \mathbb{N}, \text{ there exists } m \in \mathbb{N} \text{ with } n < m\text{”}$$

which is the same as

$$\text{“For all } n \in \mathbb{N}, \text{ there exists } m \in \mathbb{N} \text{ with } m > n\text{”}.$$

Since we know this is true, we can immediately conclude that the weaker statement in Proposition 1.2 is true as well.

In general, consider statements of the following two types:

1. For all a, there exists b such that ....
2. There exists b such that for all a, we have ....

Let's examine the difference between them in a more informal way. Think about a game with two players where Player I goes first. For the first statement to be true, it needs to be the case that no matter how Player I moves, Player II can respond in such a way so that ... happens. Notice in this scenario Player II's move can depend on Player I's move, i.e. the value of b can depend on the value of a. For the second statement to be true, it needs to be the case that Player I can make a move so brilliant that no matter how Player II responds, we have that ... happens. In this scenario, b needs to be chosen *first* without knowing a, so b can not depend on a in any way.

Finally, let's discuss one last construct in mathematical statements, which is an "if...then..." clause. We call such statements *implications*, and they naturally arise when we want quantify only over part of a set. For example, the statement

$$\text{“For all } a \in \mathbb{R}, \text{ we have } a^2 - 4 \geq 0\text{”}$$

is false because $0 \in \mathbb{R}$ and $0^2 - 4 < 0$. However, the statement

$$\text{“For all } a \in \mathbb{R} \text{ with } a \geq 2, \text{ we have } a^2 - 4 \geq 0\text{”}$$

is true. Instead of coupling the condition "$a \geq 2$" with the "for all" statement, we can instead write this statement as

$$\text{"For all } a \in \mathbb{R},\ (\text{If } a \geq 2,\ \text{then } a^2 - 4 \geq 0)\text{"}.$$

We often write this statement in shorthand by dropping the "for all" as:

$$\text{"If } a \in \mathbb{R} \text{ and } a \geq 2,\ \text{then } a^2 - 4 \geq 0.\text{"}$$

One convention, that seems quite strange initially, arises from this. Since we want to allow "if...then..." statements, we need to assign truth values to them because every mathematical statement should either be true or false. If we plug the value 3 for a into this last statement (or really past the "for all" in the penultimate statement), we arrive at the statement

$$\text{"If } 3 \geq 2,\ \text{then } 3^2 - 4 \geq 0\text{"}$$

which we naturally say is true because both the "if" part and the "then" part are true. However, it's less clear how we should assign a truth value to

$$\text{"If } 1 \geq 2,\ \text{then } 1^2 - 4 \geq 0\text{"}$$

because both the "if" part and the "then" part are false. We also have an example like

$$\text{"If } -5 \geq 2,\ \text{then } (-5)^2 - 4 \geq 0\text{"}$$

where the "if" part is false and the "then" part is true. In mathematics, we make the convention that an "if...then..." statement is false only when the "if" part is true and the "then" part is false. Thus, these last two examples we declare to be true. The reason why we do this is be consistent with the intent of the "for all" quantifier. In the example

$$\text{For all } a \in \mathbb{R},\ (\text{If } a \geq 2,\ \text{then } a^2 - 4 \geq 0),$$

we do want any value of a with $a < 2$ to have any effect at all on the truth value of the "for all" statement. Thus, we want the parenthetical statement to be true if the "if" part is false. In general, given two mathematical statements P and Q, we *define* the following.

- If P is true and Q is true, we say that "If P, then Q" is true.
- If P is true and Q is false, we say that "If P, then Q" is false.
- If P is false and Q is true, we say that "If P, then Q" is true.
- If P is false and Q is false, we say that "If P, then Q" is true.