Equivalence Relations

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1 Relations

Definition 1.1. *Let A and B be sets. A* (binary) relation *between A and B is a subset $R \subseteq A \times B$. If $A = B$, then we call a subset of $A \times A$ a* (binary) relation *on A.*

For example, let $A = \{1, 2, 3\}$ and $B = \{6, 8\}$ as above. Let

$$R = \{(1, 6), (1, 8), (3, 8)\}$$

We then have that R is a relation between A and B, although certainly not a very interesting one. However, we'll use it to illustrate a few facts. First, in a relation, it's possible for an element of A to be related to multiple elements of B, as in the case for $1 \in A$ in our example R. Also, it's possible that an element of A is related to no elements of B, as in the case of $2 \in A$ in our example R.

For a more interesting example, consider the binary relation on $\mathbb{Z}$ defined by $R = \{(a, b) \in \mathbb{Z}^2 : a < b\}$. Notice that $(4, 7) \in R$ and $(5, 5) \notin R$.

By definition, relations are sets. However, it is typically cumbersome to use set notation to write things like $(1, 6) \in R$. Instead, it usually makes much more sense to use infix notation and write $1R6$. Moreover, we can use better notation for the relation by using a symbol like $\sim$ instead of R. In this case, we would write $1 \sim 6$ instead of $(1, 6) \in \sim$ or $2 \not\sim 8$ instead of $(2, 8) \notin \sim$.

With this new notation, we give a few examples of binary relations on $\mathbb{R}$:

- Given $x, y \in \mathbb{R}$, we let $x \sim y$ if $x^2 + y^2 = 1$.
- Given $x, y \in \mathbb{R}$, we let $x \sim y$ if $x^2 + y^2 \leq 1$.
- Given $x, y \in \mathbb{R}$, we let $x \sim y$ if $x = \sin y$.
- Given $x, y \in \mathbb{R}$, we let $x \sim y$ if $y = \sin x$.

Again, notice from these examples that given $x \in \mathbb{R}$, there many 0, 1, 2, or even infinitely many $y \in \mathbb{R}$ with $x \sim y$.

If we let $A = \{0, 1\}^*$ be the set of all finite sequences of 0's and 1's, then the following are binary relations on A:

- Given $\sigma, \tau \in A$, we let $\sigma \sim \tau$ if σ and τ have the same number of 1's.
- Given $\sigma, \tau \in A$, we let $\sigma \sim \tau$ if σ occurs as a consecutive subsequence of τ (for example, we have $010 \sim 001101011$ because 010 appears in positions 5-6-7 of 001101011).

For a final example, let A be the set consisting of the 50 states. Let R be the subset of $A \times A$ consisting of those pairs of states whose second letter of their postal codes are equal. For example, we have (Iowa,California) $\in R$ and and (Iowa, Virginia) $\in R$ because the postal codes of these sets are IA, CA, VA. We also have (Minnesota, Tennessee) $\in R$ because of the postal codes MN and TN. Now (Texas, Texas) $\in R$, but there is no $a \in A$ with $a \neq$ Texas such that (Texas, a) $\in R$ because no other state has X as the second letter of its postal code. Texas stands alone.

2 Equivalence Relations

Definition 2.1. *An* equivalence relation *on a set A is a binary relation $\sim$ on A having the following three properties:*

- *$\sim$ is reflexive: $a \sim a$ for all $a \in A$.*
- *$\sim$ is symmetric: Whenever $a, b \in A$ satisfy $a \sim b$, we have $b \sim a$.*
- *$\sim$ is transitive: Whenever $a, b, c \in A$ satisfy $a \sim b$ and $b \sim c$, we have $a \sim c$.*

Consider the binary relation $\sim$ on $\mathbb{Z}$ where $a \sim b$ means that $a \leq b$. Notice that $\sim$ is reflexive because $a \leq a$ for all $a \in \mathbb{Z}$. Also, $\sim$ is transitive because if $a \leq b$ and $b \leq c$, then $a \leq c$. However, $\sim$ is not symmetric because $3 \sim 4$ but $4 \not\sim 3$. Thus, although $\sim$ satisfies two out of the three requirements, it is not an equivalence relation.

A simple example of an equivalence relation is where $A = \mathbb{R}$ and $a \sim b$ means that $|a| = |b|$. In this case, it is straightforward to check that $\sim$ is an equivalence relation.

Example 2.2. *Let A be the set $\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$, i.e. A is the set of all pairs $(a, b) \in \mathbb{Z}^2$ with $b \neq 0$. Define a relation $\sim$ on A as follows. Given $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$, we let $(a, b) \sim (c, d)$ mean $ad = bc$. We then have that $\sim$ is an equivalence relation on A.*

Proof. We check the three properties.

- Reflexive: Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Since $ab = ba$, it follows that $(a, b) \sim (a, b)$.
- Symmetric: Let $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$, and $(a, b) \sim (c, d)$. We then have that $ad = bc$. From this, we conclude that $cb = da$ so $(c, d) \sim (a, b)$.
- Transitive: Let $a, b, c, d, e, f \in \mathbb{Z}$ with $b, d, f \neq 0$ where $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. We then have that $ad = bc$ and $cf = de$. Multiplying the first equation by f we see that $adf = bcf$. Multiplying the second equation by b gives $bcf = bde$. Therefore, we know that $adf = bde$. Now $d \neq 0$ by assumption, so we may cancel it to conclude that $af = be$. It follows that $(a, b) \sim (e, f)$

Therefore, $\sim$ is an equivalence relation on A. □

Let's analyze the above situation more carefully. We have $(1, 2) \sim (2, 4)$, $(1, 2) \sim (4, 8)$, $(1, 2) \sim (-5, -10)$, etc. If we think of (a, b) as representing the fraction $\frac{a}{b}$, then the relation $(a, b) \sim (c, d)$ is saying exactly that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal. You may never have thought about equality of fractions as the result of imposing an equivalence relation on pairs of integers, but that is exactly what it is. We will be more precise about this below.

Definition 2.3. *Let $\sim$ be an equivalence relation on a set A. Given $a \in A$, we let*

$$\overline{a} = \{b \in A : a \sim b\}$$

The set $\overline{a}$ is called the equivalence class *of a.*

Some sources use the notation $[a]$ instead of $\overline{a}$. This notation helps emphasize that the equivalence class of a is a *subset* of A rather than an element of A. However, it is cumbersome notation when we begin working with equivalence classes. We will stick with our notation, although it might take a little time to get used to. Notice that by the reflexive property of $\sim$, we have that $a \in \overline{a}$ for all $a \in A$.

For example, let's return to where A is the set consisting of the 50 states and R is the subset of $A \times A$ consisting of those pairs of states whose second letter of their postal codes are equal. It's straightforward to show that R is an equivalence relation on A. We have

$$\overline{\text{Iowa}} = \{\text{California}, \text{Georgia}, \text{Iowa}, \text{Louisiana}, \text{Massachusetts}, \text{Pennsylvania}, \text{Virginia}, \text{Washington}\}$$

while

$$\overline{\text{Minnesota}} = \{\text{Indiana}, \text{Minnesota}, \text{Tennessee}\}$$

and

$$\overline{\text{Texas}} = \{\text{Texas}\}$$

Notice that each of these are sets, even in the case of $\overline{\text{Texas}}$.

For another example, suppose we are working with $A = \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ where $(a, b) \sim (c, d)$ means that $ad = bc$. As discussed above, some elements of $\overline{(1, 2)}$ are $(1, 2)$, $(2, 4)$, $(4, 8)$, $(-5, -10)$, etc. So

$$\overline{(1, 2)} = \{(1, 2), (2, 4), (4, 8), (-5, -10), \dots\}$$

Again, I want to emphasize that $\overline{(a, b)}$ is a subset of A.

The following proposition is hugely fundamental. It says that if two equivalence classes overlap, then they must in fact be equal. In other words, if $\sim$ is an equivalence on A, then the equivalence classes *partition* the set A into pieces.

Proposition 2.4. *Let $\sim$ be an equivalence relation on a set A and let $a, b \in A$. If $\overline{a} \cap \overline{b} \neq \emptyset$, then $\overline{a} = \overline{b}$.*

Proof. Suppose that $\overline{a} \cap \overline{b} \neq \emptyset$. Fix $c \in \overline{a} \cap \overline{b}$. We then have $a \sim c$ and $b \sim c$. By symmetry, we know that $c \sim b$, and using transitivity we get that $a \sim b$. Using symmetry again, we conclude that $b \sim a$.

We first show that $\overline{a} \subseteq \overline{b}$. Let $x \in \overline{a}$. We then have that $a \sim x$. Since $b \sim a$, we can use transitivity to conclude that $b \sim x$, hence $x \in \overline{b}$.

We next show that $\overline{b} \subseteq \overline{a}$. Let $x \in \overline{b}$. We then have that $b \sim x$. Since $a \sim b$, we can use transitivity to conclude that $a \sim x$, hence $x \in \overline{a}$.

Putting this together, we get that $\overline{a} = \overline{b}$. □

With that proposition in hand, we are ready for the foundational theorem about equivalence relations.

Theorem 2.5. *Let $\sim$ be an equivalence relation on a set A and let $a, b \in A$.*

1. *$a \sim b$ if and only if $\overline{a} = \overline{b}$.*
2. *$a \not\sim b$ if and only if $\overline{a} \cap \overline{b} = \emptyset$.*

Proof. We first prove 1. Suppose first that $a \sim b$. We then have that $b \in \overline{a}$. Now we know that $b \sim b$ because $\sim$ is reflexive, so $b \in \overline{b}$. Thus, $b \in \overline{a} \cap \overline{b}$, so $\overline{a} \cap \overline{b} \neq \emptyset$. By the previous proposition, we conclude that $\overline{a} = \overline{b}$.

Suppose conversely that $\overline{a} = \overline{b}$. Since $b \sim b$ because $\sim$ is reflexive, we have that $b \in \overline{b}$. Therefore, $b \in \overline{a}$ and hence $a \sim b$.

We now use everything we've shown to get 2 with little effort. Suppose that $a \not\sim b$. Since we just proved 1, it follows that $\overline{a} \neq \overline{b}$, so by the previous proposition we must have $\overline{a} \cap \overline{b} = \emptyset$. Suppose conversely that $\overline{a} \cap \overline{b} = \emptyset$. We then have $\overline{a} \neq \overline{b}$ (because $a \in \overline{a}$ so $\overline{a} \neq \emptyset$), so $a \not\sim b$ by part 1. □

Therefore, given an equivalence relation $\sim$ on a set A, the equivalence classes partition A into pieces. Working out the details in our postal code example, one can show that $\sim$ has 1 equivalence class of size 8 (namely $\overline{\text{Iowa}}$, which is the same set as $\overline{\text{California}}$ and 6 others), 3 equivalence classes of size 4, 4 equivalence classes of size 3, 7 equivalence classes of size 2, and 4 equivalence classes of size 1.

Let's revisit the example of $A = \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ where $(a, b) \sim (c, d)$ means $ad = bc$. The equivalence class of $(1, 2)$, namely the set $\overline{(1, 2)}$ is the set of all pairs of integers which are ways of representing the fraction $\frac{1}{2}$. In fact, this is how once can "construct" the rational numbers from the integers. We simply *define* the rational numbers to be the set of equivalence classes of A under $\sim$. In other words, we let

$$\frac{a}{b} = \overline{(a, b)}$$

So when we write something like

$$\frac{1}{2} = \frac{4}{8}$$

we are simply saying that

$$\overline{(1,2)} = \overline{(4,8)}$$

which is true because $(1,2) \sim (4,8)$.