Counting

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1 Arrangements, Permutations, and Combinations

Let A be a finite set with $|A| = n$. Given $k \in \mathbb{N}^+$, the set A^k is the set of all finite sequences of length n whose elements are all from A. Occasionally, especially in computer science, such a finite sequence is called a *string* over A of length k. We already know that $|A^k| = |A|^k = n^k$, so we can count the number of finite sequences of length k. For example, if $A = \{a, b, c, d\}$, then there are exactly $4^2 = 16$ many two letter strings over A. There are exactly $128^2 = 16,384$ many two character long ASCII sequences, and there are 10^7 many potential phone numbers.

Notice that in a finite sequences, we might have repetition. For example, if $A = \{1, 2, 3\}$, then $(1, 1, 3) \in A^3$ and $(3, 1, 2, 3) \in A^4$. Suppose that A is a set with $|A| = n$, and we want to count the number of sequences of length 2 where there is no repetition, i.e. we want to determine the cardinality of the set

$$B = \{(a, b) \in A^2 : a \neq b\}$$

There are (at least) two straightforward ways to do this.

- *Method 1:* We use the complement rule. Let $D = \{(a, a) : a \in A\}$ and notice that $|D| = n$ because $|A| = n$. Since $B = A^2 \backslash D$, it follows that $|B| = |A^2| - |D| = n^2 - n = n(n - 1)$.

- *Method 2:* We use a modified version of the product rule as follows. Think about constructed an element of B in two stages. First, we need to pick the first coordinate of our pair, and we have n choices here. Now once we fix the first coordinate of our pair, we have $n - 1$ choices for the second coordinate because we can choose any element of A other than the one that we chose in the first round. By making these two choices in succession, we determine an element of B, and furthermore, every element of B is obtained via a unique sequence of such choices. Therefore, we have $|B| = n(n - 1)$.

Notice that in the argument for Method 2 above, we are not directly using the Product Rule. The issue is that we can not write B in the form $B = X \times Y$ where $|X| = n$ and $|Y| = n - 1$ because the choice of second coordinates depends upon the choice of first component. For example, if $A = \{1, 2, 3\}$, then if we choose 1 as our first coordinate, then we can choose any element of $\{2, 3\}$ for the second, while if we choose 3 as our first coordinate, then we can choose any element of $\{1, 2\}$ for the second. However, the key fact is that the *number* of choices for the second coordinate is the same no matter what we choose for the first.

Suppose more generally that we are building a set of objects in stages, in such a way that a sequence of choices throughout the stages determines a unique object, and no two distinct sequences determine the same object. Suppose also that we have the following number of choices at each stage:

- There are n_1 many choices at the first stage.

- For each choice in the first stage, there are n_2 many objects to pair with it in the second stage.

- For each pair of choices in the first two stages, there are n_3 many choices to append at the third stage.

- ...
- For each sequence of choices in the first $k-1$ stages, there are n_k many choices to append at the k^{th} stage.

In this situation, there are $n_1 n_2 n_3 \cdots n_k$ many total objects in the set. The argument is similar to the argument for the Product Rule, and again we will omit a formal proof.

With this new rule in hand, we can count a new type of object.

Definition 1.1. *Let A be a finite set with $|A| = n$. A* permutation *of A is an element of A^n without repeated elements, i.e. it's a linear ordering of all n elements of A without repetition.*

For example, consider $A = \{1, 2, 3\}$. One example of permutation of A is $(3, 1, 2)$. The set of all permutations of A is:

$$\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$$

Thus, there are 6 permutations of the set $\{1, 2, 3\}$. In order to count the number of permutations of a set with n element, we use our new technique.

Proposition 1.2. *If A is a finite set with $n \in \mathbb{N}^+$ elements, then there are $n!$ many permutations of A.*

Proof. We can build a permutation of A through a sequence of choices.

- We being by choosing the first element, and we have n choices.
- Once we've chosen the first element, we have $n-1$ choices for the second because we can choose any element of A other than the one chosen in the first stage.
- Next, we have $n-2$ many choices for the third element.
- ...
- At stage $n-1$, we have chosen $n-2$ distinct elements so far, so we have 2 choices here.
- Finally, we have only choice remaining for the last position.

Since every such sequence of choices determines a permutation of A, and distinct choices given distinct permutations, it follows that there are $n(n-1)(n-2)\cdots 2 \cdot 1 = n!$ many permutations of A. □

Alternatively, we can give a recursive description of the number of permutations of a set with n-elements, and use that to derive the above result. Define $f \colon \mathbb{N}^+ \to \mathbb{N}^+$ by letting $f(n)$ be the number of permutations of $\{1, 2, \ldots, n\}$. Notice that $f(1) = 1$. Suppose that we know the value of $f(n)$. We show how to build all permutations of $\{1, 2, \ldots, n+1\}$ from the $f(n)$ many permutations of $\{1, 2, \ldots, n\}$ along with an element of the set $\{1, 2, \ldots, n, n+1\}$. Given a permutation of $\{1, 2, \ldots, n\}$ together with a number k with $1 \leq k \leq n+1$, we form a permutation of $\{1, 2, \ldots, n, n+1\}$ by taking our permutation of $\{1, 2, \ldots, n\}$, and inserting $n+1$ into the sequence in position k (and then shifting all later numbers to the right). For example, if $n = 4$ and we have the permutation $(4, 1, 3, 2)$, and we have $k = 2$, then we insert 5 into the second position to form the permutation $(4, 5, 1, 3, 2)$.

In this way, we form all permutations of $\{1, 2, \ldots, n, n+1\}$ in a unique way. More formally, if we let $\mathcal{R}_n$ is the set of all permutation $\{1, 2, \ldots, n\}$, then this rule provides a bijection from $\mathcal{R}_n \times \{1, 2, \ldots, n, n+1\}$ to $\mathcal{R}_{n+1}$. Therefore, we have $f(n+1) = (n+1) \cdot f(n)$ for all $n \in \mathbb{N}^+$. Combining this with the fact that $f(1) = 1$, we conclude that $f(n) = n!$ for all $n \in \mathbb{N}^+$.

Definition 1.3. *Let A be a finite set with $|A| = n$, and let $k \in \mathbb{N}$ with $1 \leq k \leq n$. A* partial permutation *of A of length k is an element of A^k with no repeated element. A partial permutation of length k is also called a k-*permutation *of A.*

Proposition 1.4. *If A is a finite set with $n \in \mathbb{N}^+$ elements and $k \in \mathbb{N}^+$ is such that $1 \leq k \leq n$, then there are*

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

many k-permutations of A.

Proof. The proof is the same as for permutations, except we stop after k stages. Notice that the last term in the product, corresponding to the number of choices at stage k, is $n-(k-1) = n-k+1$ because at stage k we have chosen the first $k-1$ many element. Finally, notice that

$$\begin{aligned}\frac{n!}{(n-k)!} &= \frac{n(n-1)(n-2)\cdot(n-k+1)(n-k)(n-k-1)\cdots 1}{(n-k)(n-k-1)\cdots 1} \\ &= n(n-1)(n-2)\cdots(n-k+1)\end{aligned}$$

giving the last equality. □

For example, using the standard 26-letter alphabet, there are $26 \cdot 25 \cdot 24 = \frac{26!}{23!} = 15,600$ many three-letter strings of letters having no repetition.

Notation 1.5. *If $k, n \in \mathbb{N}^+$ with $1 \leq k \leq n$, we use the notation $(n)_k$ or $P(n,k)$ for the number of k-permutations of a set with n elements, i.e. we define*

$$(n)_k = P(n,k) = \frac{n!}{(n-k)!}$$

Suppose that A and B are finite sets and $|A| = m$ and $|B| = n$.

- We claim that number of functions f with domain A and codomain B equals n^m. To see this, list the elements of A in some order as $a_1, a_2, \dots, a_m$. A function assigns a unique value in B to each a_i, so we go through that a_i in order. For a_1, we have n possible images because we can choose any element of B. Once we've chosen this, we now have n possible images for a_2. As we go along, we always have n possible images for each of the a_i. Therefore, the number of functions from A to B is $n \cdot n \cdots n = n^m$.
- Notice that if $n < m$, then there are no injective functions $f \colon A \to B$ by the Pigeonhole Principle. Suppose instead that $n \leq m$. We claim that the number of injective functions f with domain A and codomain B equals $P(n,m) = \frac{n!}{(n-m)!}$. The argument is similar to the one for general functions, but we get fewer choices as we progress through A. As above, list the elements of A in some order as $a_1, a_2, \dots, a_m$. A function assigns a unique value in B to each a_i, so we go through that a_i in order. For a_1, we have n possible images because we can choose any element of B. Once we've chosen this, we now have $n-1$ possible images for a_2 because we can choose any value of B other than the one we sent a_1 to. Then we have $n-2$ many choices for a_3, etc. Once we arrive at a_m, we have already used up $m-1$ many elements of B, so we have $n-(m-1) = n-m+1$ many choices for where to send a_m. Therefore, the number of functions from A to B is

  $$n \cdot (n-1) \cdot (n-2) \cdots (n-m+1) = \frac{n!}{(n-m)!}$$

  which is $P(n,m)$.

Suppose we ask the following question: Let $A = \{1,2,3,4,5,6,7\}$. How many element of A^4 contain the number 7 at least once? In other words, how many three digit numbers are there such that each digit is between 1 and 7 (inclusive), and 7 occurs at least once? A natural guess is that the answer is $4 \cdot 7^3$ for the follow reason:

- First, pick one of 4 positions to place the 7.
- Now we have three positions open. Going through them in order, we have 7 choices for what to put in each of these three positions.

This all looks great, but unfortunately, there is a problem. It is indeed true that such a sequence of four choices does create one of the number we are looking for. If we choose the sequence $3, 1, 5, 1$, then we obtain the number 1571. However, the sequence of choices $2, 7, 3, 4$ and the sequence of choices $1, 7, 3, 4$ both produce the same string, namely 7734.

The key idea is to count the complement. Instead of counting the number of elements of A^4 that *do* contain the number 7 at least once, we count the number of elements of A^4 that *do not* contain the number 7 at all, and subtract this amount from the total number of elements in A^4. Now since $|A| = 7$, we have that $|A^4| = 7^4$ because we have 7 choices for each of the 4 spots. To count the number of elements of A^4 that do not contain a 7, we simply notice that we have 6 choices for each of the 4 spots, so there are 6^4 of these. Therefore, by the Complement Rule, the number of elements of A^4 that do contain the number 7 at least once is $7^4 - 6^4$.

We next move on to a fundamental question that will guide a lot of our later work. Let $n \in \mathbb{N}^+$. We know that there are 2^n many subsets of $\{1, 2, \dots, n\}$. However, what if we ask how many subsets there are of a certain size? For instance, how many subsets are there of $\{1, 2, 3, 4, 5\}$ having exactly 3 elements? The intuitive idea is to make 3 choices: First, pick one of the 5 elements to go into our set. Next, put one of the 4 remaining elements to add to it. Finally, finish off the process by picking one of the 3 remaining elements. For example, if we choose the number $1, 3, 5$ then we get the set $\{1, 3, 5\}$. Thus, a natural guess is that there are $5 \cdot 4 \cdot 3$ many subsets with 3 elements. However, recall that a set has neither repetition nor order, so just as in the previous example we count the same set multiple times. For example, picking the sequence $3, 5, 1$ would also give the set $\{1, 3, 5\}$. In fact, we arrive at the set $\{1, 3, 5\}$ in the following six ways:

$$\begin{array}{ccc} 1,3,5 & 1,5,3 & 3,1,5 \\ 3,5,1 & 5,1,3 & 5,3,1 \end{array}$$

At this point, we may be tempted to throw our hands in the air as we did above. However, there is one crucial difference. In our previous example, some sequences of 4 numbers including a 7 were counted once (like 1571), some were counted twice (like 7712), and others were counted three or four times. However, in our current situation, *every* subset is counted exactly 6 times because given a set with 3 elements, we know that there are $3! = 6$ many permutations of that set (i.e. ways to arrange the elements of the set in order). The fact that we count each element 6 times means that the total number of subsets of $\{1, 2, 3, 4, 5\}$ having exactly 3 elements equals $\frac{5 \cdot 4 \cdot 3}{6} = 10$. The general principle that we are applying is the following:

Proposition 1.6 (Quotient Rule)**.** *Suppose that A is a finite set with $|A| = n$. Suppose that we arrange the elements A into groups so that each group has exactly k many members. We then have a total of $\frac{n}{k}$ many groups.*

Proof. Let ℓ be the number of groups. To obtain an element of A, we can first pick one of the ℓ groups, and then pick one of the k many elements from that group. Thus, $n = k \cdot \ell$ by the Product Rule. It follows that $\ell = \frac{n}{k}$. □

Proposition 1.7. *Let $n, k \in \mathbb{N}^+$ and with $1 \leq k \leq n$. Suppose that A is a finite set with $|A| = n$. The number of subsets of A having exactly k elements equals:*

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k! \cdot (n-k)!}$$

Proof. We generalize the above argument. We know that the number of k-permutations of A equals

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Now a k-permutation of A picks k distinct elements of A, put also assigns an order to the elements. Now every subset of A of size k is coded by exactly $k!$ many such k-permutations because we can order the subset in $k!$ many ways. Therefore, by the Quotient Rule, the number of subsets of A having exactly k elements equals

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k! \cdot (n-k)!}$$

□

Notice also that if $k = 0$, then there is one subset of any set having zero elements (namely $\emptyset$). Thus, by defining $0! = 1$, the above formula works in the case when $k = 0$ as well.

Definition 1.8. *Let $n, k \in \mathbb{N}$ and with $0 \leq k \leq n$. We define the notations $\binom{n}{k}$ and $C(n, k)$ by*

$$\binom{n}{k} = C(n,k) = \frac{n!}{k! \cdot (n-k)!}$$

We call this the number of k-combinations *of an n-element set, and pronounce $\binom{n}{k}$ as "n choose k".*

For example, the number of 5-card poker hands from a standard 52-card deck is:

$$\binom{52}{5} = \frac{52!}{5! \cdot 48!} = 2,598,960.$$

We now give a number of example of counting problems:

- Over the standard 26-letter alphabet, how many "words" of length 8 have exactly 5 consonants and 3 vowels? We build every such word in a unique way via a sequence of choices.
  - First, we pick out a subset of 3 of the 8 positions to house the vowels, and we have $\binom{8}{3}$ many possibilities.
  - Next, we pick 3 vowels in order allowing repetition to fill in these positions. Since we have 5 vowels, there are 5^3 many possibilities.
  - Finally, we pick 5 consonants in order allowing repetition to fill in remaining 5 positions. Since we have 21 consonants, there are 21^5 many possibilities.

  Since every word is uniquely determined by this sequence of choices, the number of such words is

  $$\binom{8}{3} \cdot 5^3 \cdot 21^6 = 56 \cdot 5^3 \cdot 21^5$$

- How many ways are there to seat n people around a circular table (so the only thing that matters is the relative position of people with respect to each other)? To count this, we use the Quotient Rule. We first consider each of the chairs as distinct. List the people in some order, and notice that we have n choices for where to seat the first person, then $n-1$ for where to seat the second, then $n-2$ for the third, and so forth. Thus, if the seats are distinct, then we have $n!$ many ways to seat the people. However, two such seating arrangements are equivalent if we can get one from the other via a rotation of the seats. Since there are n possible rotations, each seating arrangement occurs n times in this count, so the total number of such seatings is $\frac{n!}{n} = (n-1)!$.

- Suppose that we are in a city where all streets are straight and either east-west or north-south. Suppose that we are at one corner, and want to travel to a corner that is m blocks east and n blocks north, but we want to do it efficiently. More formally, we want to count the number of ways to get from the point $(0,0)$ to the point (m,n) where at each stage we either increase the x-coordinate by 1 or we increase the y-coordinate by 1. At first sight, it appears that we at each intersection, we have 2 choices: Either go east or go north. However, this is not really the case, because if we can east m times, then we are forced to go north the rest of the way. The idea for how to count this is that such a path is uniquely determined by a sequence of $m+n$ many E's and N's (representing east and north) having exactly m many E's. To determine such a sequence, we need only choose the positions of the m many E's, and there are
$$\binom{m+n}{m}$$
man choices. Of course, we could instead choose the positions of n many N's to count it as
$$\binom{m+n}{n}$$
which is the same number.

- How many anagrams (i.e. rearrangements of the letters) are there of MISSISSIPPI? Here is one approach. Notice that MISSISSIPPI has one M, four I's, four S's, and two P's, for a total of eleven letters. First pick the position of the M and notice that we have 11 choices. Once that is done, pick the position of the four I's and notice that this amount to picking a 4 element subset of the remaining 10 positions. There are $\binom{10}{4}$ many such choices. Once that is done, pick the position of the four S's and notice that this amount to picking a 4 element subset of the remaining 6 positions. There are $\binom{6}{4}$ many such choices. Once this is done, the position of the two P's is fixed. This gives a total number of anagrams equal to
$$11\binom{10}{4}\binom{6}{4} = 11 \cdot \frac{10!}{4! \cdot 6!} \cdot \frac{6!}{4! \cdot 2!} = \frac{11!}{4! \cdot 4! \cdot 2!} = 34,650$$
Another argument is as follows. Think of distinguishing common letters with different colors. We then have 11! many ways to rearrange the letters, but this number overcounts the numbers of anagrams. Each actual anagram comes about in $4! \cdot 4! \cdot 2!$ many ways because we can permute the currently distinct four I's amongst each other in 4! ways, we can permute the currently distinct four S's amongst each other in 4! ways, and we can permute the the currently distinct two P's amongst each other in 2! many ways. Thus, since each actual anagram is counted $4! \cdot 4! \cdot 2!$ many times in the 11! count, it follows that there are
$$\frac{11!}{4! \cdot 4! \cdot 2!} = 34,650$$
many anagrams of MISSISSIPPI.

As mentioned above, there are a total of
$$\binom{52}{5} = 2,598,960$$
many (unordered) 5-card poker hands from a standard 52-card deck. Using this, we now count the number of special hands of each type. We use the fact that each card has one of four suits (clubs, diamonds, hearts, and spades) and one of thirteen ranks (2, 3, 4, 5, 6, 7, 8, 9,10, jack, queen, king, ace). We follow the common practice of allowing the ace to be either a low card or a high card for a straight, but we do not allow "wrap around" straights such as king, ace, 2, 3, 4.

- Straight Flush: There are
$$4 \cdot 10 = 40$$
many of these because they are determined by a choice of suits and the rank of the lowest card (from ace through 10). The probability is about .00154%.

- Four of a kind: There are
$$13 \cdot 48 = 624$$
of these because we choose a rank (and take all four cards of that rank), and the choose one of the remaining 48 cards. The probability is about .0256%.

- Full House: There are
$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3,744$$
many, which can be seen by making the following sequence of choices:
  - Choose one of the 13 ranks for the three of a kind.
  - Choose 3 of the 3 suits for the three of a kind.
  - Choose one of the 12 remaining ranks for the pair.
  - Choose 2 of the 4 suits for the pair.

  The probability is about .14406%.

- Flush: There are
$$4 \cdot \binom{13}{5} = 5,148$$
many because we need to choose 1 of the 4 suits, and then 5 of the 13 ranks. However, 40 of these are actually straight flushes, so we really have 5,108 many flushes that are not stronger hands. The probability is about .19654%

- Straight: There are
$$10 \cdot 4^5 = 10,240$$
many because we need to choose the rank of the lowest card, and the suits for the five cards in increasing order of rank. However, we again have that 40 of these are straight flushes, so we really have 10,200 many straights that are not stronger hands. The probability is about .39246%.

- Three of a kind: There are
$$13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2 = 54,912$$
many, which can be seen by making the following sequence of choices:
  - Choose one of the 13 ranks for the three of a kind.
  - Choose 3 of the 3 suits for the three of a kind.
  - Choose two of the other ranks for the remaining two cards (they are different because we do not want to include full houses).
  - Choose the suit of the lower ranked card not in the three of a kind.
  - Choose the suit of the higher ranked card not in the three of a kind.

(Alternatively, we can choose the last two cards in different ranks in $48 \cdot 44$ many ways, but then we need to divide by 2 because the order of choosing these does not matter.) The probability is about 2.1128%.

- Two Pair: There are

  $$\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 44 = 123,552$$

  many, which can be seen by making the following sequence of choices:

  - Choose the two ranks for the two pairs.
  - Choose the two suits for the lower ranked pair.
  - Choose the two suits for the higher ranked pair.
  - Choose one of the 44 cards not in these two ranks.

  The probability is about 4.7539%.

- One pair: There are

  $$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240$$

  many, which can be seen by making the following sequence of choices:

  - Choose the rank for the pair.
  - Choose the two suits for the pair.
  - Choose three distinct ranks for the other three cards (which are not the same rank as the pair).
  - Choose the suit of the lowest ranked card not in the pair.
  - Choose the suit of the middle ranked card not in the pair.
  - Choose the suit of the highest ranked card not in the pair.

  The probability is about 42.257%.