Sample Written Assignment 2

Question: Explain why the method of u-substitution works.

Answer: The method of u-substitution is a technique for finding an antiderivative (or indefinite integral) of a function by making use of the Chain Rule in reverse.

Recall that the Chain Rule gives us a method for calculating the derivative of a function which is the composition of two other functions. Suppose that $F(x)$ and $g(x)$ are two functions, and we consider their composition $F(g(x))$. The Chain Rule then tells us that the derivative of this composition of functions is

$$F'(g(x)) \cdot g'(x)$$

Suppose that we are trying to find an indefinite integral $\int h(x)\, dx$. To make use of the above observations, the idea is to try to recognize our integrand $h(x)$ as being of the form $h(x) = f(g(x)) \cdot g'(x)$ for some functions $f(x)$ and $g(x)$. Suppose that we can do this, and we can also find an antiderivative $F(x)$ of $f(x)$. In this case, the Chain Rule tells us that the derivative of $F(g(x))$ is

$$F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x) = h(x)$$

so we have succeeded in finding an antiderivative of $h(x)$.

With all of this in mind, here is how the method of u-substitution works. We are facing an indefinite integral $\int h(x)\, dx$. We let $u = g(x)$ for some function $g(x)$ of our choosing. We write the notation

$$du = g'(x)\ dx$$

and mechanically substitute this into the formula by replacing dx with $\frac{du}{g'(x)}$. We then attempt to convert our resulting formula for $\frac{h(x)}{g'(x)}$ into a formula involving u's but no more x's. If we are successful, then we can recognize $\frac{h(x)}{g'(x)}$ as a function $f(u)$ in terms of u. We would then have that

$$\frac{h(x)}{g'(x)} = f(u) = f(g(x))$$

and hence

$$h(x) = f(g(x)) \cdot g'(x)$$

We now proceed to find an antiderivative $F(u)$ for the function $f(u)$. If we are successful, then the previous paragraph tells us that $F(g(x))$ is an antiderivative for $h(x)$.