# Sample Written Assignment 1

**Question (Section 5.5, #54):** Evaluate $\int_0^1 x\sqrt{1-x^4}\,dx$ by making a substitution and interpreting the resulting integral in terms of an area.

**Answer:** We begin by recalling that $x^2+y^2=1$ defines the unit circle (that is the circle of radius 1) centered at the origin. If we attempt to solve for $y$, we first notice that $y^2 = 1-x^2$, and so taking the square root we see that

$$y = \pm\sqrt{1-x^2}$$

Thus, the graph of the function $f(x) = \sqrt{1-x^2}$ gives the top half of the unit circle, and the graph of the function $g(x) = -\sqrt{1-x^2}$ gives the bottom half of the unit circle.

In attempting to evaluate the integral, we begin by making the substitution $u = x^2$ with the hope of taking out the extra $x$ out front. We then have that $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. Now when $x = 0$, we have $u = 0^2 = 0$. Similarly, when $x = 1$, we have $u = 1^2 = 1$. Therefore,

$$\begin{aligned}\int_0^1 x\sqrt{1-x^4} &= \int_0^1 \sqrt{1-(x^2)^2}\cdot x\,dx \\ &= \int_0^1 \sqrt{1-u^2}\cdot\frac{1}{2}\,du \\ &= \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du\end{aligned}$$

We now examine the integral $\int_0^1 \sqrt{1-u^2}\,du$. Since the function $f(u) = \sqrt{1-u^2}$ is nonnegative on the interval $[0,1]$, this integral is just the area of the region above the $u$-axis and below the graph of $f(u) = \sqrt{1-u^2}$ on the interval $[0,1]$. As we noted above, the function $f(u) = \sqrt{1-u^2}$ on the interval $[-1,1]$ is the graph of the top half of the unit circle. Thus, the integral

$$\int_0^1 \sqrt{1-u^2}\,du$$

is calculating exactly 1/4 of the area of the unit circle. Since the unit circle has area $\pi$, it follows that

$$\int_0^1 \sqrt{1-u^2}\,du = \frac{1}{4}\cdot\pi = \frac{\pi}{4}$$

and hence

$$\int_0^1 x\sqrt{1-x^4} = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$